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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

. . . ut proinde his paucis consideratis tota haec materia redacta sit ad puram Geometriam, quod in physicis & mechanicis unice desideratum. LEIBNIZ

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On the First Boundary-value Problem of Linear Elastostatics

M. E. GURTIN & ELI STERNBERG

1. Introduction

The complete system of field equations in the linear equilibrium theory of homogeneous isotropic elastic solids, with reference to rectangular cartesian coordinates x_i and in the usual indicial notation[1], takes the form

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{1}$$

$$\tau_{ij} = 2\mu\left[\frac{\sigma}{1-2\sigma}\,\delta_{ij}\,e_{kk} + e_{ij}\right], \tag{2}$$

$$\tau_{ij,j} + f_i = 0 \qquad (\tau_{ji} = \tau_{ij}). \tag{3}$$

These equations must hold throughout the region of space D occupied by the medium. Here $u_i(x)$, $e_{ij}(x)$, and $\tau_{ij}(x)$ are the cartesian components of the displacement vector-field[2], the strain tensor-field, and the stress tensor-field, respectively, while $f_i(x)$ denotes the components of the body-force density; the constant parameters μ and σ designate the shear modulus and POISSON's ratio, whereas δ_{ij} stands for the Kronecker delta. Elimination of the stresses and strains between (1), (2), and (3) yields the displacement equations of equilibrium, which are equivalent to the single vectorial equation[3]

$$\mu\nabla^2 \boldsymbol{u} + \frac{\mu}{1-2\sigma}\nabla\nabla\cdot\boldsymbol{u} + \boldsymbol{f} = 0 \quad \text{in } D, \tag{4}$$

where ∇ is the conventional del-operator. The present paper is concerned exclusively with the first problem of elastostatics, in which the surface displacements are prescribed on the boundary B of the region of space D. This problem evidently reduces to the determination of a vector field $\boldsymbol{u}(x)$ satisfying (4), subject to the boundary condition

$$\boldsymbol{u} = \boldsymbol{u}^*(x) \quad \text{on } B. \tag{5}$$

If D is bounded — and in the presence of appropriate regularity assumptions concerning B and $\boldsymbol{u}(x)$ — the uniqueness of the solution to the foregoing boundary-

[1] Throughout this paper Latin subscripts range over the integers (1, 2, 3) and summation over repeated subscripts is implied. Subscripts preceded by a comma indicate differentiation with respect to the corresponding cartesian coordinate.

[2] The single argument x represents the triplet of coordinates (x_1, x_2, x_3).

[3] Letters in boldface designate vectors. The symbols "$\cdot$" and "$\times$" are used to indicate scalar and vector multiplication of two vectors.

value problem is assured by the classical uniqueness theorem due to Kirchhoff [*1*][4], provided the elastic constants satisfy the inequalities

$$\mu > 0, \quad -1 < \sigma < \tfrac{1}{2}. \tag{6}$$

The traditional proof of Kirchhoff's theorem rests on the energy identity

$$\tfrac{1}{2}\int_B \boldsymbol{t}\cdot\boldsymbol{u}\,dA + \tfrac{1}{2}\int_D \boldsymbol{f}\cdot\boldsymbol{u}\,dV = \int_D W\,dV \equiv U, \tag{7}$$

where $\boldsymbol{t}$ is the surface-traction vector with components $t_i = \tau_{ij} n_j$, $\boldsymbol{n}$ being the outer unit normal of B, and U is the total strain energy. The strain-energy density W is given by

$$W = \frac{1}{2}\tau_{ij} e_{ij} = \mu\left(\frac{\sigma}{1-2\sigma} e_{ii} e_{jj} + e_{ij} e_{ij}\right), \tag{8}$$

and (6) are necessary and sufficient that W be a positive definite function of the components of strain. In view of (1), (7), (8), U may be regarded as a functional of the displacement field and admits the represention

$$U\{\boldsymbol{u}\} = \frac{\mu}{2}\int_D \left(\frac{2\sigma}{1-2\sigma} u_{i,i} u_{j,j} + u_{i,j} u_{i,j} + u_{i,j} u_{j,i}\right) dV. \tag{9}$$

The uniqueness theorem just cited may also be inferred from the principle of minimum potential energy to which it is closely related. This principle[5], within the context of the first boundary-value problem, asserts that among the displacement fields satisfying the boundary condition (5) one which also meets the field equation (4) is characterized by an absolute minimum of the functional

$$\Phi\{\boldsymbol{u}\} = U\{\boldsymbol{u}\} - \int_D \boldsymbol{f}\cdot\boldsymbol{u}\,dV. \tag{10}$$

The conventional proof of the foregoing minimum principle depends once again on an appeal to the positive definiteness of the strain-energy density and thus requires (6) to hold; further, it involves the usually tacit assumption that D is bounded.

It is natural to ask whether the positive definiteness of the strain-energy density is *necessary* for the truth of the uniqueness theorem or whether the inequalities (6) may be relaxed without loss in uniqueness. This question, though inconsequential from the point of view of applications of the theory to actual materials, is of obvious theoretical interest and, in addition, is relevant to certain considerations in nonlinear elasticity theory. Evidently, the requirement $\mu > 0$ in (6) may be replaced by $\mu \neq 0$ since Kirchhoff's argument remains valid if W is negative-definite, rather than positive-definite. Next, it follows from general results due to Browder [*3*] and Morrey [*4*] on the uniqueness of the solution to the Dirichlet problem for strongly elliptic systems of partial differential equations that (4) and (5) admit at most one (sufficiently regular) solution if

$$\mu \neq 0, \quad -\infty < \sigma < \tfrac{1}{2}, \quad 1 < \sigma < \infty. \tag{11}$$

[4] See Love [*2*], p. 170. Numbers in brackets refer to the list of publications at the end of the paper.

[5] See Love [*2*], p. 171.

Uniqueness clearly fails when $\mu=0$. The same is true when $\sigma=1$, as observed by ERICKSEN & TOUPIN [5]. Finally, it is apparent from a counter-example constructed by ERICKSEN [6] that $\frac{1}{2}<\sigma<1$ also results in lack of uniqueness so that (11) cannot be further weakened[6].

A second question arising in connection with the classical uniqueness theorem concerns its generalization to unbounded domains. Specifically, we confine our attention to regions exterior to a finite number of closed surfaces[7]. KIRCHHOFF'S argument is readily modified to accommodate such regions provided

$$u_i(x)=o(1), \quad \tau_{ij}(x)=O(r^{-2}) \quad \text{as} \quad r\to\infty, \tag{12}$$ [8]

where $r=\sqrt{x_i x_i}$. Conditions (12), however, which specify the rate of decay of the displacements and stresses at infinity, are highly artificial. A stronger and physically more useful form of the uniqueness theorem appropriate to the exterior first boundary-value problem was established recently by DUFFIN & NOLL [7], who assume merely that $\boldsymbol{u}(x)$ tends to zero uniformly as $r\to\infty$. Their contribution is an extension to elasticity theory of an earlier investigation by FINN & NOLL [8] of related issues in the theory of Stokes flows.

The proof presented in [7] is equally applicable to bounded regions and, as regards limitations on the elastic constants, presupposes merely the assumptions (11) in place of the unnecessarily restrictive hypotheses[9] (6). It thus supplies at the same time an economical scheme for obtaining directly the generalization of the uniqueness theorem discussed earlier and obviates the need for invoking the broader, but more involved, arguments contained in [3], [4]. For this reason, and in order to render the present paper sensibly self-contained, it seemed desirable to include in what follows, a slightly more compact version of the uniqueness proof due to DUFFIN & NOLL [7].

The proof referred to above, and spelled out in Section 2, makes use of the functional

$$\left.\begin{aligned} G\{\boldsymbol{u}\} &= \frac{\mu}{2}\int_D [\varkappa(\nabla\cdot\boldsymbol{u})^2+(\nabla\times\boldsymbol{u})^2]\,dV, \\ \varkappa &= \frac{2(1-\sigma)}{1-2\sigma}, \end{aligned}\right\} \tag{13}$$

which plays a role analogous to that played by the strain energy $U\{\boldsymbol{u}\}$ in the conventional proof of KIRCHHOFF'S theorem. In Section 3 we deduce a minimum principle that is the corresponding counterpart of the classical principle of minimum potential energy. We show there that among the displacement fields

[6] Note that $\sigma\neq\frac{1}{2}$ is implicit in (2) and (4); the case of the incompressible medium requires a separate treatment.

[7] Regions whose boundary extends to infinity are excluded from the present considerations.

[8] Here, as well as in the sequel, the notion of "order of magnitude" is used in its standard mathematical connotation. Thus, if $v(x)$ is a scalar or vector field defined in a neighborhood of infinity, we write $v(x)=O(r^k)$ or $v(x)=o(r^k)$ as $r\to\infty$ according as $|r^{-k}v(x)|$ remains bounded or $r^{-k}v(x)\to 0$ in this limit. All order-of-magnitude statements will henceforth be understood to refer to the limit as $r\to\infty$.

[9] This fact goes without comment in [7] which is chiefly concerned with the removal of the order-of-magnitude restrictions (12).

conforming to the boundary condition (5) one which in addition meets the field equation (4) yields an absolute minimum of

$$\Psi\{\boldsymbol{u}\} = G\{\boldsymbol{u}\} - \int_D \boldsymbol{f}\cdot\boldsymbol{u}\,dV. \tag{14}$$

This minimum principle is valid for the extended range

$$\mu > 0, \qquad -\infty < \sigma < \tfrac{1}{2}, \quad 1 < \sigma < \infty, \tag{15}$$

of the elastic constants and applies also to the exterior problem if $\boldsymbol{u}(x) = o(1)$ and provided the body forces are suitably restricted[10]. The Duffin-Noll theorem of Section 2 may be considered as a corollary of the theorem established in Section 3 if $\mu > 0$.

The connection between the two alternative minimum principles is explored in Section 4, where we show that for a bounded D the difference between the functionals $U\{\boldsymbol{u}\}$ and $G\{\boldsymbol{u}\}$ depends solely on the boundary values $\boldsymbol{u}^*(x)$, and hence is an invariant of the class of admissible displacement fields. It follows in particular that the principle of minimum potential energy continues to hold if (6) is replaced with (15), despite the fact that the strain-energy density need then no longer be positive definite.

Section 5 contains a theorem which yields lower bounds for the functional $\Psi\{\boldsymbol{u}\}$. Since the structure of $G\{\boldsymbol{u}\}$ is simpler than that of $U\{\boldsymbol{u}\}$, the results obtained in Sections 3, 4, 5 are apt to be of interest also in connection with the application of direct variational methods to the first boundary-value problem of elastostatics.

2. The uniqueness theorem of Duffin and Noll

By a *regular region of space* $D+B$ we shall mean a region of space D whose boundary B consists of a finite number of non-intersecting "closed regular surfaces", the latter term being used in the sense of Kellogg [9][11]. Thus B is necessarily piecewise smooth but may have corners and edges. Further, D need not be simply connected or bounded. If D is infinite, however, B still remains bounded. The uniqueness theorem dealt with in [7] may be stated in the subsequent somewhat more inclusive form.

Theorem 1. *Let $D+B$ be a regular region of space. There exists at most one vector field $\boldsymbol{u}(x)$, defined and twice continuously differentiable in $D+B$, which satisfies* (4) *and* (5), *provided* (11) *hold and if $\boldsymbol{u}(x) = o(1)$ in the event that D is infinite.*

The proof of the corresponding theorem in [7] may be adapted and condensed as follows. It is evidently sufficient to show that

$$\nabla^2\boldsymbol{u} + \frac{1}{1-2\sigma}\nabla\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } D, \tag{16}$$

$$\boldsymbol{u} = 0 \quad \text{on } B, \tag{17}$$

[10] The generalization of the principle of minimum potential energy to exterior domains is elementary only if the artificial regularity conditions (12) hold and provided the body forces are properly behaved at infinity.

[11] See [9], p. 112. Note that the definition of a "regular region of space" used in [9] is narrower than that employed here.

imply $\boldsymbol{u}(x)=0$ in D, if $\boldsymbol{u}(x)$ has the regularity properties stipulated above. Suppose first that D is bounded. From the divergence theorem, with the aid of (16) and elementary identities in vector analysis, follows the identity

$$\frac{\mu}{2}\int_B [\varkappa\, \boldsymbol{u}\, \nabla\cdot\boldsymbol{u} + \boldsymbol{u}\times(\nabla\times\boldsymbol{u})]\cdot\boldsymbol{n}\, dA = G\{\boldsymbol{u}\}, \tag{18}$$

where $G\{\boldsymbol{u}\}$ and $\varkappa$ are given by (13). In view of (17) and (18) one has

$$G\{\boldsymbol{u}\}=0. \tag{19}$$

Since $\varkappa>0$ according to (11) and the second of (13), we conclude from (19) and the smoothness of $\boldsymbol{u}(x)$ that $\nabla\cdot\boldsymbol{u}=\nabla\times\boldsymbol{u}=0$ in D, whence

$$\nabla^2\boldsymbol{u}=0 \quad \text{in } D. \tag{20}$$

But (20) and (17), in conjunction with the uniqueness theorem for the Dirichlet problem, assure that $\boldsymbol{u}(x)=0$ in D.

Next, consider the case of an infinite D and let $S(\varrho)$ be the surface of a sphere of radius ϱ, centered at the origin and containing B wholly in its interior. On applying the identity (18) to the region $R(\varrho)$ bounded by B and $S(\varrho)$ and using (17), (13), we arrive at

$$\int_{S(\varrho)} [\varkappa\, \boldsymbol{u}\, \nabla\cdot\boldsymbol{u} + \boldsymbol{u}\times(\nabla\times\boldsymbol{u})]\cdot\boldsymbol{n}\, dA = \int_{R(\varrho)} [\varkappa(\nabla\cdot\boldsymbol{u})^2 + (\nabla\times\boldsymbol{u})^2]\, dV. \tag{21}$$

Operating on (16) with the divergence and the curl, respectively, one confirms that[12]

$$\nabla^2\nabla\cdot\boldsymbol{u}=0, \quad \nabla^2\nabla\times\boldsymbol{u}=0 \quad \text{in } D. \tag{22}$$

We now invoke Lemma 1 of [7], which asserts that (22) and $\boldsymbol{u}(x)=o(1)$ imply

$$\nabla\cdot\boldsymbol{u}=O(r^{-2}), \quad \nabla\times\boldsymbol{u}=O(r^{-2}). \qquad (23)^{13}$$

It is clear from (23), since $\boldsymbol{u}(x)=o(1)$, that the surface integral in (21) tends to zero as $\varrho\to\infty$. Consequently the same is true of the volume integral on the right-hand side of (21), from which the conclusion $\boldsymbol{u}(x)=0$ in D follows as before in view of $\boldsymbol{u}(x)=o(1)$. This completes the proof of Theorem 1.

3. An associated minimum principle

Definition 1. *Let $D+B$ be a regular region of space. Let $\hat{\boldsymbol{u}}(x)$, defined and twice continuously differentiable in $D+B$, meet the boundary condition*

$$\hat{\boldsymbol{u}}=\boldsymbol{u}^*(x) \quad \text{on } B, \tag{24}$$

and let $\hat{\boldsymbol{u}}(x)=o(1)$ if D is infinite. The totality of all such $\hat{\boldsymbol{u}}(x)$ constitutes the class $[\hat{\boldsymbol{u}}]$ of kinematically admissible vector fields corresponding to $D+B$ and to prescribed surface values $\boldsymbol{u}^*(x)$.

We now state a minimum principle which is closely related to the uniqueness theorem of Section 2.

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[12] As is well known, any solution of (16) which is twice continuously differentiable has continuous derivatives of all orders. In this connection see also [*10*].

[13] The proof of the second of (23) is carried out in detail by Finn & Noll [*8*]; the first of (23) may be established by strictly analogous means.

Theorem 2. *Let $D+B$ be a regular region of space. Let $\boldsymbol{u}(x)$ be a vector field, defined and twice continuously differentiable in $D+B$, which satisfies* (4) *and* (5). *If D is infinite, let $\boldsymbol{u}(x)=o(1)$, $\boldsymbol{f}(x)$ be continuously differentiable in D, and*

$$\nabla\cdot\boldsymbol{f}=0, \quad \nabla\times\boldsymbol{f}=0 \quad \text{in } D. \tag{25}$$

Let $[\hat{\boldsymbol{u}}]$ be the class of kinematically admissible vector fields corresponding to $D+B$ and $\boldsymbol{u}^*(x)$. Then, assuming (15) *to hold,*

$$\Psi\{\boldsymbol{u}\}=\min_{[\hat{\boldsymbol{u}}]}\Psi\{\hat{\boldsymbol{u}}\}, \tag{26}$$

where Ψ is the functional defined in (14), *and this absolute minimum is assumed by $\Psi\{\hat{\boldsymbol{u}}\}$ only for $\hat{\boldsymbol{u}}(x)=\boldsymbol{u}(x)$ in D.*

To establish this theorem it suffices to show that

$$\Psi\{\hat{\boldsymbol{u}}\}\geqq\Psi\{\boldsymbol{u}\}, \tag{27}$$

$$\Psi\{\hat{\boldsymbol{u}}\}=\Psi\{\boldsymbol{u}\} \quad \text{if and only if} \quad \hat{\boldsymbol{u}}(x)=\boldsymbol{u}(x) \quad \text{in } D. \tag{28}$$

Defining $\boldsymbol{u}'(x)$ through

$$\boldsymbol{u}'(x)=\hat{\boldsymbol{u}}(x)-\boldsymbol{u}(x), \tag{29}$$

we consider first the case in which D is bounded. From (13), (14), and (29) follows directly

$$\Psi\{\hat{\boldsymbol{u}}\}-\Psi\{\boldsymbol{u}\}=G\{\boldsymbol{u}'\}-\int_D\boldsymbol{f}\cdot\boldsymbol{u}'\,dV+L\{\boldsymbol{u},\boldsymbol{u}'\}, \tag{30}$$

where

$$L\{\boldsymbol{u},\boldsymbol{u}'\}=\mu\int_D\left[\varkappa(\nabla\cdot\boldsymbol{u})(\nabla\cdot\boldsymbol{u}')+(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u}')\right]dV. \tag{31}$$

By hypothesis and (29) both $\boldsymbol{u}(x)$ and $\boldsymbol{u}'(x)$ are twice continuously differentiable in D and hence obey the elementary identities

$$\begin{aligned}
(\nabla\cdot\boldsymbol{u})(\nabla\cdot\boldsymbol{u}')&=\nabla\cdot(\boldsymbol{u}'\,\nabla\cdot\boldsymbol{u})-\boldsymbol{u}'\cdot\nabla(\nabla\cdot\boldsymbol{u}),\\
(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u}')&=\nabla\cdot[\boldsymbol{u}'\times(\nabla\times\boldsymbol{u})]+\boldsymbol{u}'\cdot\nabla\times(\nabla\times\boldsymbol{u}),\\
\nabla\times(\nabla\times\boldsymbol{u})&=\nabla\nabla\cdot\boldsymbol{u}-\nabla^2\boldsymbol{u}.
\end{aligned} \tag{32}$$

With the aid of (32) we may write (31) in the form

$$\begin{aligned}
L\{\boldsymbol{u},\boldsymbol{u}'\}=\mu&\int_D\nabla\cdot[\varkappa\,\boldsymbol{u}'\,\nabla\cdot\boldsymbol{u}+\boldsymbol{u}'\times(\nabla\times\boldsymbol{u})]\,dV\\
&-\mu\int_D\boldsymbol{u}'\cdot\left[\nabla^2\boldsymbol{u}+\frac{1}{1-2\sigma}\nabla\nabla\cdot\boldsymbol{u}\right]dV.
\end{aligned} \tag{33}$$

Applying the divergence theorem to the first integral on the right-hand side of (33) and bearing in mind that $\boldsymbol{u}(x)$ is a solution of (4), we reach

$$L\{\boldsymbol{u},\boldsymbol{u}'\}=\mu\int_B[\varkappa\,\boldsymbol{u}'\,\nabla\cdot\boldsymbol{u}+\boldsymbol{u}'\times(\nabla\times\boldsymbol{u})]\cdot\boldsymbol{n}\,dA+\int_D\boldsymbol{f}\cdot\boldsymbol{u}'\,dV. \tag{34}$$

Substitution from (34) into (30) yields

$$\Psi\{\hat{\boldsymbol{u}}\}-\Psi\{\boldsymbol{u}\}=G\{\boldsymbol{u}'\}+\mu\int_B[\varkappa\,\boldsymbol{u}'\,\nabla\cdot\boldsymbol{u}+\boldsymbol{u}'\times(\nabla\times\boldsymbol{u})]\cdot\boldsymbol{n}\,dA. \tag{35}$$

But, by hypothesis, $\boldsymbol{u}(x)$ and $\hat{\boldsymbol{u}}(x)$ assume the same boundary values $\boldsymbol{u}^*(x)$, so that from (29)

$$\boldsymbol{u}' = 0 \quad \text{on } B. \tag{36}$$

Therefore (35) reduces to

$$\Psi\{\hat{\boldsymbol{u}}\} - \Psi\{\boldsymbol{u}\} = G\{\boldsymbol{u}'\}. \tag{37}$$

By virtue of (13) and (15), $G\{\boldsymbol{u}'\}$ is non-negative, whence (27) is an immediate consequence of (37). Further, according to (13) and (29), $G\{\boldsymbol{u}'\}=0$ when $\hat{\boldsymbol{u}}(x)=\boldsymbol{u}(x)$. Next, $G\{\boldsymbol{u}'\}=0$, in view of (13) and (15), implies $\nabla\cdot\boldsymbol{u}'=\nabla\times\boldsymbol{u}'=0$ in D, whence by the last of (32)

$$\nabla^2\boldsymbol{u}' = 0 \quad \text{in } D. \tag{38}$$

Invoking again the uniqueness theorem for the Dirichlet problem, we draw from (38), (36) that $\boldsymbol{u}'(x)=0$ in D. Consequently, $\hat{\boldsymbol{u}}(x)=\boldsymbol{u}(x)$ if and only if $G\{\boldsymbol{u}'\}=0$ and thus (37) implies also (28). This completes the proof when D is bounded.

If D is infinite, on the other hand, the identity (35) may be applied to the bounded region $R(\varrho)$ whose boundary is $B+S(\varrho)$, where $S(\varrho)$ is once more the surface of a sphere of radius ϱ, centered at the origin and containing B in its interior. Thus and by (36), which continues to hold in the present circumstances,

$$\Psi\{\hat{\boldsymbol{u}}\} - \Psi\{\boldsymbol{u}\} = G\{\boldsymbol{u}'\} + \mu\int\limits_{S(\varrho)} [\varkappa\,\boldsymbol{u}'\,\nabla\cdot\boldsymbol{u} + \boldsymbol{u}'\times(\nabla\times\boldsymbol{u})]\cdot\boldsymbol{n}\,dA. \tag{39}$$

Recalling (13) and (14) we note that the functionals Ψ and G are now given by

$$\Psi\{\boldsymbol{u}\} = G\{\boldsymbol{u}\} - \int\limits_{R(\varrho)} \boldsymbol{f}\cdot\boldsymbol{u}\,dV, \tag{40}$$

$$G\{\boldsymbol{u}\} = \frac{\mu}{2}\int\limits_{R(\varrho)} [\varkappa(\nabla\cdot\boldsymbol{u})^2 + (\nabla\times\boldsymbol{u})^2]\,dV. \tag{41}$$

In view of the assumed smoothness of $\boldsymbol{f}(x)$ when D is infinite, we are entitled to operate on (4) with the divergence and curl, respectively, provided r is sufficiently large[14]. Making use of (25) we find in this manner that (22) and (23) hold again. Also, since $\nabla\cdot\boldsymbol{u}$ and $\nabla\times\boldsymbol{u}$ are harmonic in a neighborhood of infinity and $O(r^{-2})$, $\nabla\nabla\cdot\boldsymbol{u}$ and $\nabla\times(\nabla\times\boldsymbol{u})$ are both $O(r^{-3})$, so that the last of (32) implies that $\nabla^2\boldsymbol{u}=O(r^{-3})$. It now follows from (4) that

$$\boldsymbol{f}(x) = O(r^{-3}). \tag{42}$$

Finally, by hypothesis and (29),

$$\boldsymbol{u} = o(1), \qquad \hat{\boldsymbol{u}} = o(1), \qquad \boldsymbol{u}' = o(1). \tag{43}$$

According to (23) and (43), the surface integral in (39) tends to zero, while (23), (42), (43) and (40), (41) assure that $\Psi\{\boldsymbol{u}\}$ approaches a finite limit as $\varrho\to\infty$. Further, as is apparent from (40) to (43) and $\varkappa>0$, either $G\{\hat{\boldsymbol{u}}\}$, and hence $\Psi\{\hat{\boldsymbol{u}}\}$, tends to a finite limit or $\Psi\{\hat{\boldsymbol{u}}\}\to+\infty$ as $\varrho\to\infty$. In the latter instance (27) is trivially met. We may therefore confine our attention to admissible fields

[14] See Footnote No. 12. To demonstrate the availability of continuous third derivatives of $\boldsymbol{u}(x)$ in the present circumstances one merely needs to exhibit a particular solution of (4) which has this degree of smoothness in a neighborhood of infinity. Such a particular solution is readily constructed with the aid of spherical harmonics when $\boldsymbol{f}(x)$ meets the hypotheses of Theorem 2.

which render $G\{\boldsymbol{u}\}$ in (41) convergent. For such a choice of $\hat{\boldsymbol{u}}(x)$, (39) together with the preceding observations permit the conclusion that $G\{\boldsymbol{u}'\}$ is convergent as $\varrho\to\infty$, whence (37) remains valid if D is infinite. The conclusions (27), (28) may now be reached in precisely the same manner as before. This completes the proof of Theorem 2.

Theorem 1 becomes a corollary of Theorem 2 if[15] $\mu>0$. To demonstrate this one need merely show that $\boldsymbol{f}(x)=\boldsymbol{u}^*(x)=0$ in Theorem 2 implies $\boldsymbol{u}(x)=0$ throughout D. In these special circumstances $\hat{\boldsymbol{u}}(x)\equiv 0$ is an admissible vector field and $\Psi\{\boldsymbol{u}\}=G\{\boldsymbol{u}\}$ according to (14). Theorem 2 now implies $G\{\boldsymbol{u}\}\leqq G\{0\}$, while (13) yields $G\{0\}=0$. Hence $G\{\boldsymbol{u}\}\leqq 0$. On the other hand, $G\{\boldsymbol{u}\}\geqq 0$ as a consequence of (13). Therefore $G\{\boldsymbol{u}\}=0$, which is possible only if $\boldsymbol{u}(x)=0$ in D since at present $\boldsymbol{u}=\boldsymbol{u}^*(x)=0$ on B.

4. Relation to the principle of minimum potential energy

The minimum principle asserted in Theorem 2 passes over into the classical principle of minimum potential energy (for the first boundary-value problem) if the functional Ψ in (26) is replaced with the potential energy Φ, defined by (10), provided (6) are required to hold in place of (15) and on the assumption that D is bounded. In view of (10) and (14), the transition from Ψ to Φ is, in turn, effected by substituting U for G in (14). Both minimum principles are equivalent to the same boundary-value problem (4), (5), Equation (4) in either case being the Euler equation of the corresponding variational principle. This observation suggests a connection between the functionals G and U which we establish presently.

Theorem 3. *Let $D+B$ be a bounded regular region of space. Let $\boldsymbol{u}(x)$ and $\hat{\boldsymbol{u}}(x)$ be defined as in Theorem 2, and set*

$$H\{\boldsymbol{u}\}=G\{\boldsymbol{u}\}-U\{\boldsymbol{u}\}, \tag{44}$$

where G and U are the functionals given by (13) *and* (9). *Then*

$$H\{\hat{\boldsymbol{u}}\}=H\{\boldsymbol{u}\}=k \qquad \text{(constant)}, \tag{45}$$

i.e. *$H\{\hat{\boldsymbol{u}}\}$ is an invariant of the class of kinematically admissible vector fields* $[\hat{\boldsymbol{u}}]$.

With a view toward a proof of Theorem 3 we observe first on the basis of (13) that $G\{\hat{\boldsymbol{u}}\}$ may be written as

$$G\{\hat{\boldsymbol{u}}\}=\frac{\mu}{2}\int_D(\varkappa\,\hat{u}_{i,i}\hat{u}_{j,j}+\hat{u}_{i,j}\hat{u}_{i,j}-\hat{u}_{i,j}\hat{u}_{j,i})\,dV. \tag{46}$$

From (9), (44), (46) and elementary manipulation follows

$$H\{\hat{\boldsymbol{u}}\}=\mu\int_D(\hat{u}_{i,i}\hat{u}_j-\hat{u}_{j,i}\hat{u}_i)_{,j}\,dV. \tag{47}$$

Using the divergence theorem in conjunction with (47), we are led to

$$H\{\hat{\boldsymbol{u}}\}=\mu\int_B h\,dA, \tag{48}$$

where

$$h(x)=(\hat{u}_{i,i}\hat{u}_j-\hat{u}_{j,i}\hat{u}_i)\,n_j. \tag{49}$$

[15] When $\mu<0$, Theorem 2 gives way to the analogous maximum principle, from which Theorem 1 may be deduced.

We now complete the argument by showing that the values of $h(x)$ on B depend solely on the boundary values $\boldsymbol{u}^*(x)$, which are common to all members of $[\hat{\boldsymbol{u}}]$. It is clearly sufficient to confirm this fact for an arbitrary regular point of B. Let Q be such a point, whence B has a uniquely defined tangent plane at Q. For convenience, choose the rectangular coordinate frame in such a way that the origin is at Q and the plane $x_3=0$ coincides with the tangent plane of B at Q, the x_3-axis pointing in the direction of the *inner* normal of B at Q. For this choice of coordinates (49) yields

$$h(x) = \hat{u}_1 \frac{\partial \hat{u}_3}{\partial x_1} + \hat{u}_2 \frac{\partial \hat{u}_3}{\partial x_2} - \hat{u}_3 \left(\frac{\partial \hat{u}_1}{\partial x_1} + \frac{\partial \hat{u}_2}{\partial x_2} \right). \tag{50}$$

In a neighborhood of Q the boundary B admits the parametrization

$$x_1 = \xi_1, \qquad x_2 = \xi_2, \qquad x_3 = \varphi(\xi_1, \xi_2), \tag{51}$$

where $\varphi(\xi_1, \xi_2)$ is continuously differentiable and

$$\varphi(0,0) = 0, \qquad \left.\frac{\partial \varphi}{\partial \xi_\alpha}\right|_{(0,0)} = 0 \qquad (\alpha = 1, 2). \tag{52}$$

Now

$$\hat{\boldsymbol{u}}(x_1, x_2, x_3) = \boldsymbol{u}^*(\xi_1, \xi_2) \quad \text{on } B, \tag{53}$$

in the neighborhood of Q under consideration. Also, by (51), (52), (53)

$$\left.\frac{\partial \hat{\boldsymbol{u}}(x)}{\partial x_\alpha}\right|_{(0,0,0)} = \left.\frac{\partial \boldsymbol{u}^*(\xi)}{\partial \xi_\alpha}\right|_{(0,0)} \qquad (\alpha = 1, 2). \tag{54}$$

Hence, from (50) and (54),

$$h(0,0,0) = \left[u_1^* \frac{\partial u_3^*}{\partial \xi_1} + u_2^* \frac{\partial u_3^*}{\partial \xi_2} - u_3^* \left(\frac{\partial u_1^*}{\partial \xi_1} + \frac{\partial u_2^*}{\partial \xi_2} \right) \right]_{(0,0)}. \tag{55}$$

Therefore $h(x)$ at Q depends exclusively on the boundary values $\boldsymbol{u}^*$ of $\hat{\boldsymbol{u}}$ and is independent of its interior values. Recalling (48) we conclude that (45) is bound to hold.

Theorem 2 is evidently an immediate consequence of Theorem 3 and of the principle of minimum potential energy, if D is bounded and the elastic constants conform to (6). Conversely, when D is bounded and with limitation to the first boundary-value problem, the principle of minimum potential energy follows from Theorem 2 and Theorem 3 even if μ and σ satisfy merely the weaker restrictions (15). The classical minimum principle is thus seen to be valid also when $-\infty < \sigma \leq -1$ or $1 < \sigma < \infty$, in which case the strain-energy density is no longer positive definite.

5. A lower-bound theorem

The minimum principle of Theorem 2 supplies upper bounds for the functional $\Psi\{\boldsymbol{u}\}$ and hence, if D is bounded, also for the potential energy

$$\Phi\{\boldsymbol{u}\} = \Psi\{\boldsymbol{u}\} - k, \tag{56}$$

once the constant k has been computed corresponding to a particular set of surface data $\boldsymbol{u}^*(x)$. We now seek to determine lower bounds for $\Psi\{\boldsymbol{u}\}$. This may be accomplished with the aid of a familiar scheme going back to an idea of TREFFTZ [*11*] that has given rise to a large number of more general

investigations concerned with the estimation of quadratic functionals[16]. In preparation for the theorem to follow we introduce the subsequent auxiliary definition, which is a counterpart of Definition 1.

Definition 2. *Let $D+B$ be a regular region of space. Let $\tilde{\boldsymbol{u}}(x)$, defined and twice continuously differentiable in $D+B$, meet* (4), *as well as the integral condition*

$$\int_B [\varkappa(\boldsymbol{u}^* - \tilde{\boldsymbol{u}})\,\nabla\cdot\tilde{\boldsymbol{u}} + (\boldsymbol{u}^* - \tilde{\boldsymbol{u}})\times(\nabla\times\tilde{\boldsymbol{u}})]\cdot\boldsymbol{n}\,dA = 0, \tag{57}$$

where $\varkappa$ is given by the second of (13). *If D is infinite, let $\tilde{\boldsymbol{u}}(x) = o(1)$. The totality of all such $\tilde{\boldsymbol{u}}(x)$ constitutes the class $[\tilde{\boldsymbol{u}}]$ of statically admissible vector fields corresponding to $D+B$, $\boldsymbol{f}(x)$, and to prescribed surface values $\boldsymbol{u}^*(x)$.*

Theorem 4. *Let $D+B$, $\boldsymbol{u}(x)$, and $\boldsymbol{f}(x)$ have the same properties as in Theorem 2. Let $[\tilde{\boldsymbol{u}}]$ be the class of statically admissible vector fields corresponding to $D+B$, $\boldsymbol{f}(x)$, and $\boldsymbol{u}^*(x)$. Then, assuming* (15) *to hold,*

$$\Psi\{\boldsymbol{u}\} = \max_{[\tilde{\boldsymbol{u}}]} \Psi\{\tilde{\boldsymbol{u}}\}, \tag{58}$$

where Ψ is the functional defined in (14).

In proving Theorem 4, one may follow closely the procedure adopted earlier to establish Theorem 2. At present we need to show that

$$\Psi\{\tilde{\boldsymbol{u}}\} \leqq \Psi\{\boldsymbol{u}\}, \tag{59}$$

$$\Psi\{\tilde{\boldsymbol{u}}\} = \Psi\{\boldsymbol{u}\} \quad \text{if} \quad \tilde{\boldsymbol{u}}(x) = \boldsymbol{u}(x) \quad \text{in } D. \tag{60}$$ [17]

Define $\boldsymbol{u}''(x)$ through

$$\boldsymbol{u}''(x) = \boldsymbol{u}(x) - \tilde{\boldsymbol{u}}(x) \tag{61}$$

and assume initially that D is bounded. The same reasoning that led to (35) now yields

$$\Psi\{\boldsymbol{u}\} - \Psi\{\tilde{\boldsymbol{u}}\} = G\{\boldsymbol{u}''\} + \mu\int_B [\varkappa\,\boldsymbol{u}''\,\nabla\cdot\tilde{\boldsymbol{u}} + \boldsymbol{u}''\times(\nabla\times\tilde{\boldsymbol{u}})]\cdot\boldsymbol{n}\,dA. \tag{62}$$

Since the surface integral in (62) vanishes by virtue of (61), (57), and (5), we have

$$\Psi\{\boldsymbol{u}\} - \Psi\{\tilde{\boldsymbol{u}}\} = G\{\boldsymbol{u}''\}, \tag{63}$$

from which (59), (60) follow at once because of (13) and (15).

In the event that D is infinite, (63) may evidently be replaced with

$$\Psi\{\boldsymbol{u}\} - \Psi\{\tilde{\boldsymbol{u}}\} = G\{\boldsymbol{u}''\} + \mu\int_{S(\varrho)} [\varkappa\,\boldsymbol{u}''\,\nabla\cdot\tilde{\boldsymbol{u}} + \boldsymbol{u}''\times(\nabla\times\tilde{\boldsymbol{u}})]\cdot\boldsymbol{n}\,dA, \tag{64}$$

in which Ψ and G are currently defined by (40), (41), while $S(\varrho)$, as well as $R(\varrho)$, retain their previous meaning.

By hypothesis and from (61),

$$\boldsymbol{u} = o(1), \quad \hat{\boldsymbol{u}} = o(1), \quad \boldsymbol{u}'' = o(1). \tag{65}$$

Also, arguing as in the analogous part of the proof of Theorem 2, we confirm once again the validity of the estimates (23), (42), and in addition arrive at

$$\nabla\cdot\tilde{\boldsymbol{u}} = O(r^{-2}), \quad \nabla\times\boldsymbol{u} = O(r^{-2}). \tag{66}$$

[16] We shall not attempt to list here even the most significant contributions to the literature on this subject. Many of the pertinent references may be found at the end of a monograph by SYNGE [*12*].

[17] Note that, in contrast to (28), the equality in (59) may hold even if $\tilde{\boldsymbol{u}}(x) = \boldsymbol{u}(x)$ fails to hold throughout D.

According to (66) and the last of (65), the surface integral in (64) tends to zero as $\varrho \to \infty$. Similarly, using (41), (23), (61) and (66), we conclude that $G\{\boldsymbol{u}''\}$ approaches a finite limit as $\varrho \to \infty$. Thus (63), and consequently (59), (60), continue to hold when D is infinite. The proof of Theorem 4 is now complete.

Note added in Proof. The authors are indebted to Dr. R. J. KNOPS, who recently drew their attention to a paper by KELVIN [Sir WILLIAM THOMSON, *On the reflexion and refraction of light*, Philosophical Magazine, **26** (1888), 414]. This investigation is concerned in part with the relaxation of the usual conditions (6) for the stability of the unstrained equilibrium configuration of an isotropic elastic solid. Specifically, KELVIN shows that the strain energy $U\{\boldsymbol{u}\}$, given in (9), coincides with the functional $G\{\boldsymbol{u}\}$, defined in (13), provided the region occupied by the solid is finite and if the surface displacements vanish. He observes further that $G\{\boldsymbol{u}\}$ is positive provided the velocities of irrotational and equivoluminal waves appropriate to such a solid are both real, *i.e.* if the inequalities (15) hold true. The preceding observations of KELVIN are readily extended to yield a proof of the generalized uniqueness Theorem 1 for the case of a *bounded* domain. Also, the introduction of the functional $G\{\boldsymbol{u}\}$, which in the present paper is attributed to DUFFIN & NOLL [7], should be credited to KELVIN.

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Linear Functional Electromagnetic Constitutive Relations and Plane Waves in a Hemihedral Isotropic Material

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1. Introduction

In any Lorentz frame of reference (x^i, t), where x^i $(i=1, 2, 3)$ are rectangular Cartesian spatial coordinates and t is the time, MAXWELL'S equations in a *stationary*, polarizable and magnetizable medium can be written in the form

$$\frac{\partial}{\partial t}\int_{\mathscr{A}} \boldsymbol{B}\cdot d\boldsymbol{a} + \oint_{\mathscr{L}} \boldsymbol{E}\cdot d\boldsymbol{l} = 0, \tag{1.1}$$

$$\oint_{\mathscr{A}} \boldsymbol{B}\cdot d\boldsymbol{a} = 0, \tag{1.2}$$

$$\oint_{\mathscr{L}} \boldsymbol{H}\cdot d\boldsymbol{l} - \frac{\partial}{\partial t}\int_{\mathscr{A}} \boldsymbol{D}\cdot d\boldsymbol{a} = \int_{\mathscr{A}} \boldsymbol{J}_{\mathrm{F}}\cdot d\boldsymbol{a}, \tag{1.3}$$

$$\oint \boldsymbol{D}\cdot d\boldsymbol{a} = \int Q_{\mathrm{F}}\, dv, \tag{1.4}$$

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}, \qquad \boldsymbol{H} = \mu_0^{-1}\boldsymbol{B} - \boldsymbol{M}, \tag{1.5}$$

where $\boldsymbol{B}$ is the *magnetic flux density*, $\boldsymbol{E}$ the *electric field*, $\boldsymbol{H}$ the *magnetic intensity*, $\boldsymbol{D}$ the *electric displacement*, $\boldsymbol{J}_{\mathrm{F}}$ the *density of free current*, Q_{F} the *density of free charge*, $\boldsymbol{P}$ the *polarization*, $\boldsymbol{M}$ the *magnetization*, and ε_0 and μ_0 are fundamental constants whose product $\varepsilon_0\mu_0 = c^{-2}$ where c is the velocity of light in vacuum. $d\boldsymbol{l}$ denotes an element of an arbitrary closed contour $\mathscr{L}$ in space and $d\boldsymbol{a}$ an element of a surface $\mathscr{A}$ bounded by $\mathscr{L}$.

The system of equations (1.1) to (1.5) is underdetermined, and to obtain a determinate system of equations amongst the twenty-two unknown field components $(B_i, E_i, H_i, D_i, P_i, M_i, J_{\mathrm{F}}, Q_{\mathrm{F}})$ it is necessary to supplement the basic Maxwell equations with certain *constitutive relations*. The form of these constitutive relations depends on the nature of the *material medium* in which the electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$ resides. The great mathematical variety of these electromagnetic constitutive relations makes possible a great variety of physical phenomena embraced by and consistent with MAXWELL'S framework of equations (1.1) to (1.5).

The simplest of all media, *vacuum*, is characterized by the constitutive relations

$$\boldsymbol{J}_{\mathrm{F}} = Q_{\mathrm{F}} = \boldsymbol{M} = \boldsymbol{P} = 0. \tag{1.6}$$

The next simplest medium is the rigid, linear, stationary, non-conducting dielectric, for which the constitutive relations

$$Q_F = \boldsymbol{J}_F = 0, \qquad \boldsymbol{D} = \boldsymbol{\epsilon} \cdot \boldsymbol{E}, \quad \boldsymbol{B} = \boldsymbol{\mu} \cdot \boldsymbol{H}, \tag{1.7}$$

were given by MAXWELL [1873, § 784]. Here $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are constant second order tensors, proportional to the unit tensor if the medium be isotropic. As is known, constitutive relations of this simple type do not account for the observed absorption and dispersion of electromagnetic waves in non-conductors, nor does MAXWELL's device of including a linear law of conduction, $\boldsymbol{J}_F = \boldsymbol{C} \cdot \boldsymbol{E}$, replacing $(1.7)_2$, suffice to account for the observed magnitude of the "dielectric losses" when a material is placed in a variable electric field (*Cf.* WHITEHEAD [1927, Lecture 1]).

HOPKINSON [1877], following a suggestion of MAXWELL, proposed a constitutive relation for the electric displacement in a dielectric having the form

$$\boldsymbol{D}(t) = \varepsilon \boldsymbol{E}(t) + \int_{-\infty}^{t} \varphi(t-\tau)\, \boldsymbol{E}(\tau)\, d\tau, \tag{1.8}$$

where the function $\varphi(u)$, $u \geqq 0$, is a decreasing function of u. By suitable adjustment of the *memory function* φ, HOPKINSON was able to correlate his data on the residual charge of Leyden jars. In proposing (1.8), HOPKINSON was guided by the earlier mechanical constitutive relation of BOLTZMANN [1874] in which the torque and twist of a wire are related by a formula identical with (1.8).

MAXWELL (*cf.* RAYLEIGH [1899]; WHITTAKER [1951]) initiated also the method which enjoys current favor of deriving electromagnetic constitutive relations from a molecular model of a material medium and the laws of mechanics. Elaborate theories of electromagnetic absorption and dispersion of this general type were developed by DRUDE [1893, 1902], VOIGT [1899], and many others. A detailed and modern theory of electromagnetic constitutive relations as derived from the dynamics of an ionic crystal lattice, including an appropriate application of quantum mechanics, is given in the book by BORN & HUANG [1954].

VOLTERRA [1912] extended the HOPKINSON's relation (1.8) to the nonlinear, anisotropic, and magnetic case. VOLTERRA's most general expression of the idea took the form of the equations:

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{x},t) &= \boldsymbol{\epsilon} \cdot \boldsymbol{E}(\boldsymbol{x},t) + \underset{-\infty}{\overset{t}{\boldsymbol{F}}}[\boldsymbol{E}(\boldsymbol{x},\tau)], \\ \boldsymbol{B}(\boldsymbol{x},t) &= \boldsymbol{\mu} \cdot \boldsymbol{H}(\boldsymbol{x},t) + \underset{-\infty}{\overset{t}{\boldsymbol{\Phi}}}[\boldsymbol{H}(\boldsymbol{x},\tau)], \end{aligned} \tag{1.9}$$

where $(1.9)_1$ reduces to (1.8) if the functional $\boldsymbol{F}$ is linear, isotropic, and satisfies certain other conditions of which we shall say more later. VOLTERRA's theory was developed further by GRAFFI [1927, 1928].

VOLTERRA's paper, though it treats the general anisotropic case and expresses a very general view, makes the traditional *a priori* separation of electric and magnetic effects characteristic of MAXWELL's simple relations (1.7). That this separation is inadequate to account for numerous known phenomena such as optical activity and the rotation of the plane of polarization of light by a strong magnetic field (Faraday effect) should have been apparent from the earlier

theory of Voigt cited above. In this paper we wish to explore some of the more obvious consequences of electromagnetic constitutive relations having the general form

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{x}, t) &= \underset{-\infty}{\overset{t}{\boldsymbol{\Phi}}}[\boldsymbol{E}(\boldsymbol{x}, \tau), \boldsymbol{B}(\boldsymbol{x}, \tau)], \\ \boldsymbol{H}(\boldsymbol{x}, t) &= \underset{-\infty}{\overset{t}{\boldsymbol{\Psi}}}[\boldsymbol{E}(\boldsymbol{x}, \tau), \boldsymbol{B}(\boldsymbol{x}, \tau)]. \end{aligned} \tag{1.10}$$

We shall show that if Volterra's relations (1.9) and the relations (1.10) are both linearized, then, for *holohedral isotropic materials*, they coincide so that there is, for these materials, a separation of effects; however, we shall show that the phenomenon of optical activity, which does not occur in holohedral isotropic materials, depends in an essential way upon the appearance of $\boldsymbol{B}$ in $(1.10)_1$ or of $\boldsymbol{E}$ in $(1.10)_2$. We shall show that, so far as the dispersion and absorption of plane electromagnetic waves is concerned, *linear* functional constitutive relations of the form (1.10) which are consistent with Volterra's *principe du cycle fermé* (explained in § 2) lead to dispersion formulae similar to those obtained by Born & Huang from a molecular model. The simple and direct mathematical reasoning we apply to the relations (1.10) should have application in any theory of the electromagnetic field in stationary matter which leads ultimately to relations consistent with (1.10). The physical idea expressed by these relations is simply that, in a stationary medium, the values of the electric field $\boldsymbol{E}(\boldsymbol{x}, \tau)$ and the magnetic flux density $\boldsymbol{B}(\boldsymbol{x}, \tau)$ for all times $\tau \leqq t$ preceding the instant t uniquely determine the values of the electric displacement $\boldsymbol{D}(\boldsymbol{x}, t)$ and the magnetic intensity $\boldsymbol{H}(\boldsymbol{x}, t)$ at the time t.

2. Restrictions imposed on the constitutive relations by material symmetry

If the material considered has some symmetry, the constitutive relations (1.10) must be form invariant under the group of transformations $\{\boldsymbol{S}\}$ describing the symmetry. For solids, the group of *material symmetry transformations* $\{\boldsymbol{S}\}$ is either the full orthogonal group or a subgroup of it. Let $\boldsymbol{S} = \|s_{ij}\|$ be a generic transformation of this group. Then the functionals on the right-hand side of (1.10) must satisfy the equations

$$\begin{aligned} \underset{-\infty}{\overset{t}{\Phi_i}}[\overline{\boldsymbol{E}}(\tau), \overline{\boldsymbol{B}}(\tau)] &= s_{ij} \underset{-\infty}{\overset{t}{\Phi_j}}[\boldsymbol{E}(\tau), \boldsymbol{B}(\tau)], \\ \underset{-\infty}{\overset{t}{\Psi_i}}[\overline{\boldsymbol{E}}(\tau), \overline{\boldsymbol{B}}(\tau)] &= \det \boldsymbol{S}\, s_{ij} \underset{-\infty}{\overset{t}{\Psi_j}}[\boldsymbol{E}(\tau), \boldsymbol{B}(\tau)], \end{aligned} \tag{2.1}$$

where

$$\overline{E}_i = s_{ij} E_j, \qquad \overline{B}_i = \det \boldsymbol{S}\, s_{ij} B_j, \tag{2.2}$$

for every element $\boldsymbol{S}$ of $\{\boldsymbol{S}\}$. The factor $\det \boldsymbol{S} = \pm 1$ appears in $(2.1)_2$ and $(2.2)_2$ because, as one can recall from elementary electromagnetic theory, the invariance of Maxwell's equations requires that $\boldsymbol{D}$ and $\boldsymbol{E}$ transform as absolute vectors under time-independent orthogonal transformations if the current and charge are absolute quantities, while $\boldsymbol{B}$ and $\boldsymbol{H}$ must transform as axial vectors under this group.

In this paper, we shall treat only the case where $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are *linear* functionals. This will exclude the more interesting non-linear phenomena such as the Faraday effect, but we choose to dispense with the simpler linear theory first. In the literature on dielectrics (*cf.*, *e.g.*, WHITEHEAD [1927]), linearity of the functional $\mathbf{\Phi}$ is justified by what is called the *principle of superposition*. Actually, we shall make the more restrictive assumption that $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are linear functionals having the form

$$\begin{aligned} \boldsymbol{D}(t) &= \sum_{\nu=0}^{p} \boldsymbol{a}_\nu \cdot \boldsymbol{E}^{(\nu)}(t) + \sum_{\nu=0}^{p} \boldsymbol{c}_\nu \cdot \boldsymbol{B}^{(\nu)}(t) + \int_{-\infty}^{t} \boldsymbol{\varphi}_1(t,\tau)\cdot \boldsymbol{E}(\tau)\,d\tau + \int_{-\infty}^{t} \boldsymbol{\varphi}_2(t,\tau)\cdot \boldsymbol{B}(\tau)\,d\tau, \\ \boldsymbol{H}(t) &= \sum_{\nu=0}^{p} \boldsymbol{d}_\nu \cdot \boldsymbol{E}^{(\nu)}(t) + \sum_{\nu=0}^{p} \boldsymbol{b} \cdot \boldsymbol{B}^{(\nu)}(t) + \int_{-\infty}^{t} \boldsymbol{\psi}_1(t,\tau)\cdot \boldsymbol{B}(\tau)\,d\tau + \int_{-\infty}^{t} \boldsymbol{\psi}_2(t,\tau)\cdot \boldsymbol{E}(\tau)\,d\tau, \quad (2.3) \\ & A^{(\nu)} = \frac{d^\nu A}{dt^\nu}, \end{aligned}$$

where $\boldsymbol{a}_\nu$, $\boldsymbol{b}_\nu$, $\boldsymbol{c}_\nu$, and $\boldsymbol{d}_\nu$ are constant tensors and the kernels $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\psi}_1$, and $\boldsymbol{\psi}_2$ are continuous tensor functions of t and τ such that each satisfies an order condition of the form

$$\boldsymbol{\varphi}_1(t,\tau) < \frac{C}{(t-\tau)^{1+\varepsilon}}, \qquad \varepsilon > 0. \tag{2.4}$$

Thus, in VOLTERRA's terminology [1930, Chap. I], $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are linear functionals with order of continuity p having t as an exceptional point. Now VOLTERRA [1912] has proven that, with the order condition (2.4), $\boldsymbol{D}(t)$ and $\boldsymbol{H}(t)$ will be periodic functions of t whenever $\boldsymbol{E}(t)$ and $\boldsymbol{B}(t)$ are periodic functions (principe du cycle fermé) *if and only if* the kernel functions $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\psi}_2$ depend on the two variables t and τ only through their difference $t-\tau$. Again, we shall delimit the class of materials under consideration by assuming (2.4) and that $\boldsymbol{D}$ and $\boldsymbol{H}$ are periodic when $\boldsymbol{E}$ and $\boldsymbol{B}$ are periodic. This latter assumption rules out something like an "ageing" or deterioration of the material. Heredity of the nature we assume here is also called *invariable heredity*.

For the linear functionals (2.3), the restrictions of symmetry (2.1) are satisfied if and only if the second-order tensors which completely determine the functional relations (2.3) satisfy the restrictions:

$$\begin{gathered} \boldsymbol{S}\,\boldsymbol{a}_\nu \boldsymbol{S}^{-1} = \boldsymbol{a}_\nu, \quad \boldsymbol{S}\,\boldsymbol{b}_\nu \boldsymbol{S}^{-1} = \boldsymbol{b}_\nu, \quad \boldsymbol{S}\boldsymbol{\varphi}_1 \boldsymbol{S}^{-1} = \boldsymbol{\varphi}_1, \quad \boldsymbol{S}\boldsymbol{\psi}_1 \boldsymbol{S}^{-1} = \boldsymbol{\psi}_1 \\ (\det \boldsymbol{S})\,\boldsymbol{S}\,\boldsymbol{c}_\nu \boldsymbol{S}^{-1} = \boldsymbol{c}_\nu, \qquad (\det \boldsymbol{S})\,\boldsymbol{S}\,\boldsymbol{d}_\nu \boldsymbol{S}^{-1} = \boldsymbol{d}_\nu, \\ (\det \boldsymbol{S})\,\boldsymbol{S}\boldsymbol{\varphi}_2 \boldsymbol{S}^{-1} = \boldsymbol{\varphi}_2, \qquad (\det \boldsymbol{S})\,\boldsymbol{S}\boldsymbol{\psi}_2 \boldsymbol{S}^{-1} = \boldsymbol{\psi}_2, \end{gathered} \tag{2.5}$$

for each element $\boldsymbol{S}$ of the group of orthogonal transformations $\{\boldsymbol{S}\}$. In other words, $\boldsymbol{a}_\nu$, $\boldsymbol{b}_\nu$, $\boldsymbol{\varphi}_1$, and $\boldsymbol{\psi}_1$ must be invariant absolute second-order tensors under transformations of $\{\boldsymbol{S}\}$ and $\boldsymbol{c}_\nu$, $\boldsymbol{d}_\nu$, $\boldsymbol{\varphi}_2$, and $\boldsymbol{\psi}_2$ must be invariant *axial* second-order tensors under transformations of $\{\boldsymbol{S}\}$. One can, in principle, determine the most general solution of the invariant theoretic problem posed by (2.5) for any group of orthogonal transformations by methods which are known.

If $\{\boldsymbol{S}\}$ is the full orthogonal group, the material is called *holohedral isotropic*, there are no invariant axial second-order tensors, and the absolute invariant second-order tensors are all proportional to the unit matrix. Thus, for holohedral isotropic materials, the constitutive relations (2.3) must reduce to VOLTERRA's

[1912] form

$$\begin{aligned} \boldsymbol{D}(t) &= \sum_{\nu=0}^{p} a_\nu \boldsymbol{E}^{(\nu)}(t) + \int_{-\infty}^{t} \varphi_1(t-\tau)\, \boldsymbol{E}(\tau)\, d\tau, \\ \boldsymbol{H}(t) &= \sum_{\nu=0}^{p} b_\nu \boldsymbol{B}^{(\nu)}(t) + \int_{-\infty}^{t} \psi_1(t-\tau)\, \boldsymbol{B}(\tau)\, d\tau. \end{aligned} \tag{2.6}$$

Thus we see that linearity *and* holohedral isotropy are sufficient conditions to eliminate $\boldsymbol{B}$ from $(1.10)_1$ and $\boldsymbol{E}$ from $(1.10)_2$, as had been *assumed* possible by Volterra for a material of general symmetry.

If $\{\boldsymbol{S}\}$ is the *proper* orthogonal group, consisting of all orthogonal transformations with positive determinant, then all of the tensors in (2.5) may be arbitrary multiples of the unit matrix and the constitutive relations (2.3) reduce to the form

$$\begin{aligned} \boldsymbol{D}(t) &= \sum_{\nu=0}^{p} a_\nu \boldsymbol{E}^{(\nu)}(t) + \sum_{\nu=0}^{p} c_\nu \boldsymbol{B}^{(\nu)}(t) + \int_{-\infty}^{t} [\varphi_1(t-\tau)\, \boldsymbol{E}(\tau) + \varphi_2(t-\tau)\, \boldsymbol{B}(\tau)]\, d\tau, \\ \boldsymbol{H}(t) &= \sum_{\nu=0}^{p} d_\nu \boldsymbol{E}^{(\nu)}(t) + \sum_{\nu=0}^{p} b_\nu \boldsymbol{B}^{(\nu)}(t) + \int_{-\infty}^{t} [\psi_1(t-\tau)\, \boldsymbol{B}(\tau) + \psi_2(t-\tau)\, \boldsymbol{E}(\tau)]\, d\tau. \end{aligned} \tag{2.7}$$

A material with this symmetry is called *hemihedral isotropic*. A sugar solution containing unequal amounts of right- and left-handed sugar molecules may serve as an example of a hemihedral isotropic material. The examples (2.6) and (2.7) suffice to indicate how one may proceed to obtain explicit representations for the linear functionals (2.3) for the case of an arbitrary symmetry group $\{\boldsymbol{S}\}$.

3. The propagation of an infinite plane electromagnetic wave in a hemihedral isotropic material

An infinite plane progressive vector wave $\boldsymbol{A}(\boldsymbol{x}, t)$ is a time-dependent vector field having the form

$$\boldsymbol{A}(\boldsymbol{x}, t) = \mathscr{R}\, \boldsymbol{a}\, e^{i(k\boldsymbol{n}\cdot\boldsymbol{x} - \omega t)}, \tag{3.1}$$

where $\boldsymbol{a} = \boldsymbol{a}^+ + i\boldsymbol{a}^-$ ($i = \sqrt{-1}$, $\boldsymbol{a}^+$ and $\boldsymbol{a}^-$ real vectors) is a complex vector called the *amplitude* of $\boldsymbol{A}$, $k = k^+ + ik^-$, $k^+ > 0$, is the *complex wave number*, $\boldsymbol{n}$ is a real unit vector called the *direction of propagation*, and $\omega > 0$ is the *angular frequency* measured in radians per unit time. The symbol $\mathscr{R}$ denotes the real part of the complex quantity placed after it.

For each fixed value of $\boldsymbol{x}$, $\boldsymbol{A}$ is a periodic vector function of time and the locus of $\boldsymbol{A}$ is an ellipse or one of its degenerate forms, a circle, or a straight line. The major axes of the ellipse, the radius of the circle, or the length and direction of the straight line are determined by the complex vector amplitude $\boldsymbol{a}$. If $\boldsymbol{a}\cdot\boldsymbol{a} = \boldsymbol{a}^+\cdot\boldsymbol{a}^+ - \boldsymbol{a}^-\cdot\boldsymbol{a}^- + 2i\boldsymbol{a}^+\cdot\boldsymbol{a}^- = 0$, the real and imaginary parts of $\boldsymbol{a}$ are perpendicular to each other and of equal length, and the locus of $\boldsymbol{A}$ is a circle. If $\boldsymbol{a}^* = \boldsymbol{a}^+ - i\boldsymbol{a}^-$ denotes the complex conjugate of $\boldsymbol{a}$, and $\boldsymbol{a}^* \times \boldsymbol{a} = 2i\boldsymbol{a}^+ \times \boldsymbol{a}^- = 0$, the real and imaginary parts of $\boldsymbol{a}$ are parallel and the locus of $\boldsymbol{A}$ is a straight line. In all other cases, the locus of $\boldsymbol{A}$ is an ellipse.

In electromagnetic theory (*cf.* Condon & Odishaw 1958, Chap. VI), if the electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ is a plane wave, then its amplitude $\boldsymbol{e}$ determines its *polarization*. An electromagnetic plane wave is said to be *circularly*, *linearly*, or *elliptically polarized* according to whether the locus of $\boldsymbol{E}$ is a circle, a straight

line, or an ellipse*. Circularly and elliptically polarized waves are further classified as *left-handed* or *right-handed* according to the rule

$$\boldsymbol{a}^+ \cdot (\boldsymbol{a}^- \times \boldsymbol{n}) = [\boldsymbol{a}^+, \boldsymbol{a}^-, \boldsymbol{n}] \begin{array}{l} > 0 \rightarrow \text{right-handed}, \\ < 0 \rightarrow \text{left-handed}. \end{array}$$

If $\boldsymbol{a} \cdot \boldsymbol{n} = 0$, the wave is called *transverse*, and if $\boldsymbol{a} \times \boldsymbol{n} = 0$, the wave is called *longitudinal*. All other waves will be called *skew*.

The field equations which follow from MAXWELL's equations (1.1) to (1.4) are

$$\begin{aligned} &\frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{curl} \boldsymbol{E} = 0, && \operatorname{div} \boldsymbol{B} = 0, \\ &\operatorname{curl} \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J}_{\mathrm{F}}, && \operatorname{div} \boldsymbol{D} = Q_{\mathrm{F}}. \end{aligned} \tag{3.2}$$

In this section, we shall consider plane electromagnetic waves in a non-conducting hemihedral isotropic medium for which $\boldsymbol{J}_{\mathrm{F}} = Q_{\mathrm{F}} = 0$, and for which the linear functional constitutive relations (2.7) are valid. Thus we seek plane wave solutions,

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}, t) &= \mathscr{R}\, \mathbf{e}\, e^{i(k\boldsymbol{n}\cdot\boldsymbol{x} - \omega t)}, \\ \boldsymbol{B}(\boldsymbol{x}, t) &= \mathscr{R}\, \mathbf{b}\, e^{i(k\boldsymbol{n}\cdot\boldsymbol{x} - \omega t)}, \\ \boldsymbol{D}(\boldsymbol{x}, t) &= \mathscr{R}\, \mathbf{d}\, e^{i(k\boldsymbol{n}\cdot\boldsymbol{x} - \omega t)}, \\ \boldsymbol{H}(\boldsymbol{x}, t) &= \mathscr{R}\, \mathbf{h}\, e^{i(k\boldsymbol{n}\cdot\boldsymbol{x} - \omega t)}, \end{aligned} \tag{3.3}$$

of the system of integral and differential equations (2.7) and (3.2).

Introducing vector fields of the form (3.3) into (2.7) and (3.2), we find that they will constitute a solution if and only if the amplitudes $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{d}$, and $\boldsymbol{h}$, the wave number k, and the frequency ω satisfy jointly the system of equations

$$\boldsymbol{n} \cdot \boldsymbol{b} = 0, \qquad \boldsymbol{n} \cdot \boldsymbol{d} = 0, \tag{3.4}$$

$$k\,\boldsymbol{n} \times \boldsymbol{e} - \omega\, \boldsymbol{b} = 0, \tag{3.5}$$

$$k\,\boldsymbol{n} \times \boldsymbol{h} + \omega\, \boldsymbol{d} = 0, \tag{3.6}$$

$$\boldsymbol{d} = \alpha(\omega)\, \boldsymbol{e} + \gamma(\omega)\, \boldsymbol{b}, \tag{3.7}$$

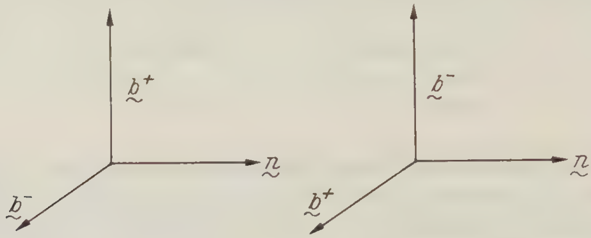
$$\boldsymbol{h} = \beta(\omega)\, \boldsymbol{b} + \delta(\omega)\, \boldsymbol{e}, \tag{3.8}$$

where

$$\begin{aligned} \alpha(\omega) &\equiv \sum_{\nu=0}^{p} a_\nu (-i\omega)^\nu + \int_0^\infty \varphi_1(u)\, e^{i\omega u}\, du, \\ \gamma(\omega) &\equiv \sum_{\nu=0}^{p} c_\nu (-i\omega)^\nu + \int_0^\infty \varphi_2(u)\, e^{i\omega u}\, du, \\ \beta(\omega) &\equiv \sum_{\nu=0}^{p} b_\nu (-i\omega)^\nu + \int_0^\infty \psi_1(u)\, e^{i\omega u}\, du, \\ \delta(\omega) &\equiv \sum_{\nu=0}^{p} d_\nu (-i\omega)^\nu + \int_0^\infty \psi_2(u)\, e^{i\omega u}\, du. \end{aligned} \tag{3.9}$$

* Linearly polarized electromagnetic waves are also called *plane* polarized waves, but this seems to overwork the word "plane".

From (3.4), (3.7), and (3.8) it follows that $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{H}$ are transverse. Since we assume $k^+>0$, $\omega>0$, so that the wave travels in the positive direction of $\boldsymbol{n}$, equations (3.4) will be satisfied as a consequence of (3.5) and (3.6). Thus it is sufficient that we consider the four homogeneous vector equations (3.5) to (3.8) in the four unknowns $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{h}$, and $\boldsymbol{d}$. In order for there to exist non-trivial solutions, the determinant $\Delta(k,\omega)$ of the 12×12 matrix of coefficients must vanish. Thus the complex wave number k is some function of the angular frequency ω, and this function, whatever it may be, is called the *dispersion*



Right-circularly polarized $\underset{\sim}{B}$ Left-circularly polarized $\underset{\sim}{B}$

Fig. 1

relation. To determine this relation and other properties of the solution we proceed as follows. First eliminate $\boldsymbol{d}$ and $\boldsymbol{h}$ from (3.6) using (3.7) and (3.8) to get

$$k\,\boldsymbol{n}\times(\beta\,\boldsymbol{b}+\delta\boldsymbol{e})+\omega(\alpha\,\boldsymbol{e}+\gamma\,\boldsymbol{b})=0. \tag{3.10}$$

Now eliminate $\boldsymbol{e}$ from (3.10) using (3.5) and $\boldsymbol{n}\cdot\boldsymbol{e}=0$ to get

$$\left(\beta-\alpha\frac{\omega^2}{k^2}\right)\boldsymbol{n}\times\boldsymbol{b}+\left(\frac{\omega}{k}\right)(\delta+\gamma)\,\boldsymbol{b}=0. \tag{3.11}$$

The scalar product of (3.11) with $\boldsymbol{b}$ then yields the condition

$$(\delta+\gamma)\,\boldsymbol{b}\cdot\boldsymbol{b}=0. \tag{3.12}$$

Thus *either the coefficient* $\delta+\gamma=0$ *or the magnetic flux* $\boldsymbol{B}$ must be *circularly polarized*. But if $\boldsymbol{B}$ is circularly polarized so also are $\boldsymbol{E}$, $\boldsymbol{D}$, and $\boldsymbol{H}$, as follows from (3.5) to (3.8).

Let us assume that $(\gamma+\delta)\neq0$ so that $\boldsymbol{B}$ is circularly polarized. Since it is also transverse, $\boldsymbol{b}^+$, $\boldsymbol{b}^-$, and $\boldsymbol{n}$ are mutually orthogonal, so that one of the two geometrical relations in Fig. 1 holds.

From Figure 1 it is easy to see that

$$\boldsymbol{n}\times\boldsymbol{b}=\begin{cases}-i\,\boldsymbol{b} & \text{for right-circular polarization,}\\ +i\,\boldsymbol{b} & \text{for left-circular polarization.}\end{cases}$$

Using this result and (3.11) we then see that the dispersion relation is

$$\frac{k^2}{\omega^2}\pm\frac{i\,k}{\omega}\,\frac{\gamma+\delta}{\beta}-\frac{\alpha}{\beta}=0, \tag{3.13}$$

where the upper sign in the linear term holds for right-circular polarization and the lower sign for left-circular polarization.

Thus far we have imposed no restrictions on the real constants a_ν, b_ν, c_ν, d_ν, or on the real memory functions $\varphi_1(u)$, $\varphi_2(u)$, $\psi_1(u)$, and $\psi_2(u)$, other than the order condition (2.4). We shall now assume that $k^+ > 0$ and $k^- \geqq 0$. It has already been pointed out that the first assumption implies that the wave travels in the positive direction of $\boldsymbol{n}$. Subject to $k^+ > 0$, the second assumption implies that a wave of frequency ω and small amplitude does not build up, resulting in an unstable situation. We may then express the root of (3.13) which satisfies $k^+ > 0$, $k^- \geqq 0$ in the form

$$\frac{k}{\omega} = \sqrt{\frac{\alpha}{\beta} - \left(\frac{\gamma+\delta}{2\beta}\right)^2} \mp i\,\frac{\gamma+\delta}{2\beta}, \tag{3.14}$$

where the upper sign in the bracket holds for right-circularly polarized $\boldsymbol{B}$ and the lower sign for left-circularly polarized $\boldsymbol{B}$, and the square root taken in each case denotes the determination for which $k^+ > 0$, $k^- \geqq 0$.

The *wave length* is then given by $\lambda = 2\pi/k^+$, the *speed* of the wave by $v = \omega/k^+$, and the *absorption coefficient* by k^-. From (3.14) we see that, unless the quantity

$$\vartheta(\omega) \equiv \mathscr{I}\left(\frac{\gamma+\delta}{\beta}\right) \tag{3.15}$$

vanishes, the speeds of the right and left-circularly polarized waves will be different so that the medium is *optically active* (see DRUDE [1902]). Unless the quantity

$$\varkappa(\omega) \equiv \mathscr{R}\left(\frac{\gamma+\delta}{\beta}\right) \tag{3.16}$$

vanishes, the right and left-circularly polarized waves will have different absorption coefficients, as has been observed in some real materials.

4. Comparison with the Born-Huang theory

Beginning with a model of an ionic solid consisting of a set of charged mass points, the dynamics of which may be treated either classically or quantum mechanically, BORN & HUANG [1954, 1, Chap. VII] derive a relation between $\boldsymbol{d}$ and $\boldsymbol{e}$ having the form

$$\boldsymbol{d} = \boldsymbol{\epsilon}(\boldsymbol{k}, \omega) \cdot \boldsymbol{e}, \qquad \boldsymbol{k} \equiv k\,\boldsymbol{n}, \tag{4.1}$$

where the *dielectric tensor* $\boldsymbol{\epsilon}$ is Hermitian,

$$\varepsilon_{ij}(\boldsymbol{k}, \omega) = \varepsilon^*_{ji}(\boldsymbol{k}, \omega), \tag{4.2}$$

and satisfies also the relations

$$\varepsilon_{ij}(\boldsymbol{k}, \omega) = \varepsilon^*_{ij}(-\boldsymbol{k}, \omega). \tag{4.3}$$

Thus if we put

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^+ + i\,\boldsymbol{\epsilon}^-,$$

the conditions (4.2) and (4.3) are satisfied if and only if

$$\begin{aligned} \varepsilon^+_{ij}(\boldsymbol{k}, \omega) &= \varepsilon^+_{ji}(\boldsymbol{k}, \omega), \\ \varepsilon^-_{ij}(\boldsymbol{k}, \omega) &= -\,\varepsilon^-_{ji}(\boldsymbol{k}, \omega), \\ \varepsilon^+_{ij}(\boldsymbol{k}, \omega) &= \varepsilon^+_{ij}(-\boldsymbol{k}, \omega), \\ \varepsilon^-_{ij}(\boldsymbol{k}, \omega) &= -\,\varepsilon^-_{ij}(-\boldsymbol{k}, \omega). \end{aligned} \tag{4.4}$$

Now from our equations (3.5) and (3.7), we get a relation like (4.1) with

$$\varepsilon_{ij}(\boldsymbol{k},\omega) = \alpha\,\delta_{ij} + \frac{\gamma k}{\omega}\, e_{ikj}\, n_k, \tag{4.5}$$

where e_{ijk} is the completely antisymmetric axial tensor with $e_{123}=1$. Thus we get

$$\begin{aligned} \varepsilon_{ij}^{+} &= \alpha^{+}\,\delta_{ij} + \frac{(\gamma k)^{+}}{\omega}\, e_{ikj}\, n_k, \\ \varepsilon_{ij}^{-} &= \alpha^{-}\,\delta_{ij} + \frac{(\gamma k)^{-}}{\omega}\, e_{ikj}\, n_k. \end{aligned} \tag{4.6}$$

The four conditions (4.4) are satisfied if and only if

$$(\gamma k)^{+} = 0, \qquad \alpha^{-} = 0. \tag{4.7}$$

It is obvious that these conditions are not met by an arbitrary assignment of the constants $a_\nu, b_\nu, c_\nu, d_\nu$, and the memory functions $\varphi_1, \varphi_2, \psi_1$, and ψ_2. In the special case of our relations for which $\alpha(\omega)$, $\beta(\omega)$ are real and positive and the functions $\gamma(\omega)$ and $\delta(\omega)$ are pure imaginary, k is real, there is no absorption, and the conditions (4.7) are both satisfied.

Further comparison of the two theories is not warranted since even the partial overlapping of results outlined above is hardly more than accidental. In the Born-Huang treatment, absorption is represented by a transfer of electromagnetic energy to mechanical energy of lattice vibrations, while in the simple phenomenological theory presented here, any motion or deformation of the medium has been ignored; moreover, we have assumed that the medium is homogeneous, while the treatment of Born & Huang is based on a highly inhomogeneous material medium, namely, a system of mass points.

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Flow of an Incompressible Fluid through an Oscillating Staggered Cascade

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Contents

1. Introduction

Under certain operating conditions the stator and rotor blades of axial-flow compressors may show the phenomenon of flutter. These oscillations may cause the breaking of the blades. One important task in the study of flutter is calculation of the unsteady velocity field and of the forces acting on the oscillating blades. For the single airfoil this problem has already been solved satisfactorily by various scientists, *e.g.* [*1, 2, 3*] and others in numerous investigations. Owing to the complexity of the problem in the case of cascades of airfoils, various authors on this subject have obtained solutions only under strongly simplifying assumptions.

The annular channel where the stator or rotor blades operate is assumed to be of height small in comparison with the radius, so that radial dependence is negligible. The plane cascade resulting from development of a blade row is assumed to consist of thin, slightly cambered profiles which in general are only slightly loaded if there is only a steady flow. It is supposed that the fluid is incompressible and nonviscous. Some papers reporting on the investigations of these problems are the following.

BILLINGTON [*4*], SÖHNGEN [*5*] and CHANG & CHU [*6*] evaluated the air forces acting on a vertical cascade of arbitrary spacing when the profiles make small harmonic oscillations in phase. They used different methods. LILLEY [*7*] and LEGENDRE [*8*] treated the case where adjacent airfoils oscillate harmonically 180° out of phase. The problem of a single airfoil executing small harmonic oscillations and being centered between two parallel walls is equivalent to the latter and was solved by TIMMAN [*9*]. MENDELSON & CARROLL [*10*] succeeded

in obtaining some formulae for lift and moment, if the line profiles of a vertical cascade with arbitrary spacing oscillate with a phase shift of 0°, 90° or 180°. Woods [*11*] by using conformal transformation and Nickel [*12*] by solving the appropriate integral equations generalized the investigations, extending them to arbitrary, unsteady flows through a cascade of line profiles, the pressure distributions being equal on adjacent blades. Söhngen [*13*] evaluated the forces on a loaded, plane cascade of staggered plates with small spacing, if independently of each other the line profiles execute small harmonic oscillations. He succeeded in verifying that a loaded blade row — free to move only in bending — can flutter. In his thesis Sisto [*14*] produced approximate calculations for cascades the profiles of which oscillate with an arbitrary phase shift.

It is of great importance that Söhngen & Quick [*15*, *16*] pointed out that only in the case of small frequencies and small Mach numbers of the basic flow the fluid may be regarded as incompressible. This fact is in full contrast with the single airfoil in unbounded space, since there are generalized characteristic modes of the gas in the annular channel. Therefore the potential equation dominating the incompressible flow problems must be replaced by the wave equation; of course, that makes all the calculations more difficult. The forces have been evaluated only for a cascade with very small spacing [*16*]. In the future it will be necessary to concentrate all the investigations on unsteady subsonic (or supersonic) flows through cascades of profiles.

Nevertheless, the author believes that the following problem is of some interest, because it generalizes those which were cited in [*1*] to [*14*]. The problem is the following:

Find the complex perturbation velocity field induced by the harmonic oscillations of the line profiles in a staggered cascade, and develop formulae for the unsteady forces acting on the separate blades. The basic flow will be assumed parallel to the line profiles in their mean positions. The N blades of the original blade row may move independently of each other with small amplitudes of deflection.

We shall solve this problem in the following way: In Chapter 2 we shall state the formula for the required field of velocities in the complex z-plane. This field will be induced by the distribution of vortices on the line profiles and in their wakes. The boundary conditions will result in a system of singular integral equations for the vortex densities. According to Kelvin's theorem and the assumption, quite usual in a linear flow theory, that the free vortices remain at the place where they arise, we shall be able to evaluate the densities of the vortices in the wakes, to within one constant factor each, ε_k, in the case of harmonic time-dependence. In the course of our later calculations the ε_k will be taken as known. Then we can evaluate the downwash on the profiles induced by the trailing vortices. The system of integral equations for the determination of the vortex densities on the profiles can easily be transformed into a new system which corresponds to a steady complex velocity field that is irrotational in the wakes.

Unfortunately, direct solution of the boundary-value problem by means of the integral equations cannot be obtained in the general case. For this reason, in Chapter 3 the z-plane will be mapped conformally onto the ζ-plane in such

a way that the staggered cascade turns into a horizontal one. We shall observe that the edges of the profiles in the z-plane will not be transformed into the edges of the corresponding profiles in the ζ-plane.

In Chapter 3 we split the transferred boundary values in the ζ-plane into an even and an odd part, regarding the two sides of the profiles. The boundary-value problems arising may be formulated under the Hilbert type. They will be solved by methods described in the book of MUSKHELISHVILI [17]. We shall obtain a unique solution if we take into consideration the fact that the perturbation velocities must vanish far upstream and remain bounded at the trailing edges of the line profiles, due to KUTTA's condition in the original z-plane.

In order to obtain the unsteady perturbation flow required originally, the N as yet unknown amplitudes ε_k of the trailing vortices must be determined by the Kelvin circulation theorem. This condition results in a system of linear equations for these N unknowns, which will be separated by means of a suitable Fourier transform.

If the perturbation velocity field is known, the lift and moment acting on each blade can be stated. In Chapter 5 we shall derive formulae for the Fourier transforms of these quantities. We wish to emphasize that these formulae are only of theoretical interest; for practical calculations they are too complicated. At this point approximate methods must be applied, and a beginning should be furnished by the evaluation of the integrals stated in the Appendix. We shall not carry the work further, since it would require some lengthy investigations.

2. Reduction of the Unsteady Flow Problem to a Steady One

Let us agree upon the notation for the parameters which specify the geometry of the cascade.

a) $2c =$ chord length of the profiles,
b) $a =$ distance of the leading edges of the profiles,
c) $T = a/c =$ spacing of the cascade,
d) $\lambda =$ angle of stagger, $-\pi/2 < \lambda \leqq +\pi/2$.

In the plane of the cascade we introduce the dimensional Cartesian co-ordinates x_0, y_0 and the complex co-ordinate $z_0 = x_0 + i \cdot y_0$ and at the same time a non-dimensional Cartesian system $0xy$. The connexion between the co-ordinate systems is given by $z_0 = c \cdot z$. The individual line profiles of the cascade are given analytically by $z_0 = c \cdot x' + i \cdot a\, m\, e^{-i\lambda}$ or $z_0 = c \cdot z_m[x']$, m an integer, and $z_m[x'] = x' + i \cdot T m e^{-i\lambda}$, $-1 \leqq x' \leqq +1$. Parallel to its positive x_0-axis this staggered cascade of line profiles is approached by an incompressible and inviscid fluid at the rate U_0. The individual points $z_m[x']$ of the profiles are assumed to move from their mean positions in harmonic deflections of small amplitude. Let the harmonic time-dependence be given by the frequency ν. In accordance with the linear perturbation theory we assume the profiles to remain in their mean positions given above and only the prescribed velocities normal to the blades to vary. The deflection velocity of the point $z_m[x']$ on the m^{th} blade in the direction of the positive y-axis at the time t is denoted by $U_0 \cdot g_m(x') \cdot \exp(2\pi\nu j t) = U_0 \cdot g(z_m[x']) \cdot \exp(2\pi\nu j t)$, $-1 \leqq x' \leqq +1$. Since the cascade is thought of as resulting from the development of a blade row

with N blades,

(1) $$g_m(x') = g_k(x') \quad \text{for } m = k + n \cdot N$$

with $k = 0, 1, \ldots, N-1$ and n an arbitrary integer.

Here we have the following problem: Find a complex perturbation velocity field with the same harmonic time-dependence,

$$U_0 \cdot W(z) \cdot \exp(2\pi \nu j t) = U_0 \cdot [U(x, y) - i \cdot V(x, y)] \cdot \exp(2\pi \nu j t),$$

such that the total velocity field

(2) $$\mathfrak{w}(z, t) = U_0 + U_0 \times \times W(z) \cdot \exp(2\pi \nu j t)$$

satisfies the boundary conditions

(3) $$\mathrm{Im}_i\{\mathfrak{w}(z_m[x'], t)\} = -U_0 \cdot g_m(x') \cdot \exp(2\pi \nu j t)$$

on the blades $z = z_m[x']$, $-1 < x' < +1$, m an integer, and the Kutta condition

(4) $$\mathfrak{w}(z, t) = O(1) \quad \text{as} \quad z \to z_m[1].$$

In addition the Kelvin theorem, which we shall soon formulate precisely, must be taken into consideration.

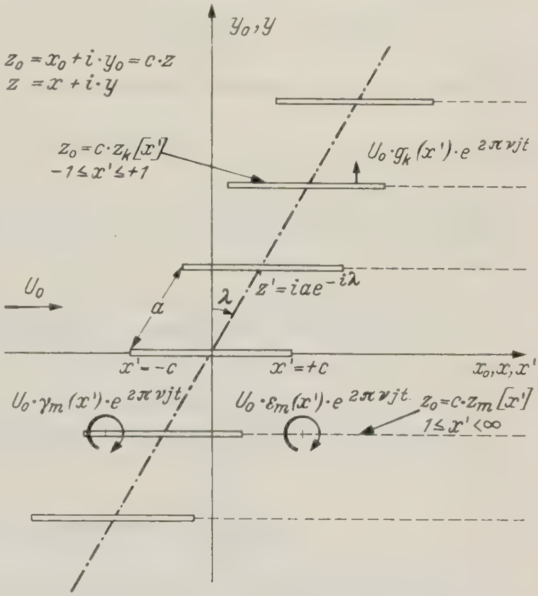

Fig. 1. Geometry of the Cascade

(The two imaginary units i and j must be distinguished strictly. They must not be confused, which means in particular that $i\,j = -1$ is not valid. In all formulae containing the factor $\exp(2\pi \nu j t)$ we mean that the real part with respect to j, Re_j must be taken.) We shall write the time-dependent factor in the form $\exp(2\pi \nu j t) = \exp(j \omega s)$ with $s = U_0 \cdot t/c$ and $\omega = 2\pi \nu c/U_0$.

Since the prescribed normal velocities are equal on either side of an individual blade, we try to solve our problem by distributing vortices on the blades. For the densities of these "profile vortices" we put $U_0 \cdot \gamma_m(x') \cdot \exp(j \omega s)$ for $-1 \leqq x' \leqq +1$, m any integer, and demand $\gamma_m(x') = \gamma_k(x')$ for $m = k + n \cdot N$, $k = 0, 1, \ldots, N-1$, n an arbitrary integer, due to equation (1). The total circulation around one profile is time dependent and will be written in the form

(5) $$U_0 \cdot c \cdot \Gamma_m(s) = U_0 \cdot c \cdot \int_{-1}^{+1} \gamma_m(x')\, dx' \cdot \exp(j \omega s).$$

Any change in the total circulation around the m^{th} profile causes free vortices to be shed from the trailing edge $z' = c \cdot z_m[1]$ according to KELVIN'S circulation theorem. We assume that the free vortices are carried away with the basic flow at the rate U_0 into the wake, given by $z' = c \cdot z_m[x']$ with $+1 \leqq x' < \infty$, and that their strength remains constant. If the harmonic motion of the profiles

has lasted for a very long time, the free vortices will have formed a plane vortex sheet in the wake of the m^{th} blade. Since the terms for perturbation velocities, vortex densities, *etc.* always contain the factor $\exp(j\,\omega\,s)$, we may cancel it in all those equations and restrict our investigations to the amplitudes. In order to build up our complex velocity field we start with vortices of strength $c\cdot U_0\times\delta_k(x')\,dx'$, which are placed at the points $z'=c\cdot z_k[x']+c\cdot i\,T\cdot N\cdot e^{-i\lambda}$, n an arbitrary integer. The velocity field induced by such a series of vortices is known to be [*13*]

$$dW_k^0(z,x')=\delta_k(x')\,d\,x'\cdot\frac{i\,e^{i\lambda}}{2T\,N}\,\mathrm{ctnh}\left\{\frac{\pi\,e^{i\lambda}}{T\,N}\left(z-z_k[x']\right)\right\}. \tag{6}$$

However, far upstream this field does not vanish, for

$$\lim_{x\to-\infty} d\,W_k^0(z,x')=-\,\delta_k(x')\,d\,x'\cdot\frac{i\,e^{i\lambda}}{2T\,N}\,. \tag{7}$$

Our cascade being thought of as resulting from a development, the field far upstream must vanish. Thus we add the homogenous field $\delta_k(x')\,d\,x'\cdot i\,e^{i\lambda}/2T\,N$ to the field of equation (6). If $\gamma_k(x')$ is the strength of the "profile vortices", and if $\varepsilon_k(x')$ denotes the strength of the free vortices in the wake of the k^{th} blade, we integrate over the chord length and the wake, respectively, sum up the N vortex systems, and obtain the total complex velocity field as follows:

$$\begin{aligned}W(z)=&\frac{i\,e^{i\lambda}}{2T\,N}\sum_{k=0}^{N-1}\int_{-1}^{+1}\gamma_k(x')\cdot\left[\mathrm{ctnh}\left(\frac{\pi\,e^{i\lambda}}{T\,N}\left(z-z_k[x']\right)\right)+1\right]d\,x'+\\&+\frac{i\,e^{i\lambda}}{2T\,N}\sum_{k=0}^{N-1}\int_{1}^{\infty}\varepsilon_k(x')\left[\mathrm{ctnh}\left(\frac{\pi\,e^{i\lambda}}{T\,N}\left(z-z_k[x']\right)\right)+1\right]d\,x'.\end{aligned} \tag{8}$$

This field vanishes as $x=\mathrm{Re}_i\,z\to-\infty$ and remains bounded as $x\to+\infty$, if $\sum_{k=0}^{N-1}\int_1^{\infty}\varepsilon_k(x')\,d\,x'$ is bounded.

The Kelvin circulation theorem requires that the relation

$$\frac{d}{dt}\,U_0\cdot c\cdot\int_{-1}^{+1}\gamma_k(x')\,d\,x'\cdot\exp(j\,\omega\,s)+U_0^2\cdot\varepsilon_k(1)\cdot\exp(j\,\omega\,s)=0 \tag{9}$$

shall hold for $k=0,1,\dots,N-1$, and, because the free vortices flow off with the basic stream, we have

$$U_0\cdot\varepsilon_k(x')\cdot\exp(j\,\omega\,s)=U_0\cdot\varepsilon_k(1)\cdot\exp\left[j\,\omega\,(s-x'+1)\right] \tag{10}$$

for $x'\geqq 1$. In the case of harmonic time dependence equations (9) and (10) imply that

$$\varepsilon_k(x')=-\,j\,\omega\cdot\exp(j\,\omega-j\,\omega\,x')\cdot\int_{-1}^{+1}\gamma_k(\xi')\,d\,\xi' \tag{11}$$

for $x'\geqq 1$ and $k=0,1,\dots,N-1$. For conciseness we put

$$\varepsilon_k=-\,j\,\omega\cdot e^{j\omega}\cdot\int_{-1}^{+1}\gamma_k(\xi')\,d\,\xi'. \tag{12}$$

Now the densities of the free vortices are known except for the N constants ε_k. We insert the terms for $\varepsilon_k(x')$ into (8), transform ctnh $\cdots + 1$, and obtain

$$W(z) = \frac{i\,e^{i\lambda}}{TN}\sum_{k=0}^{N-1}\int_{-1}^{+1}\gamma_k(x')\cdot\frac{\exp\frac{2\pi e^{i\lambda}z}{TN}\cdot dx'}{\exp\frac{2\pi e^{i\lambda}z}{TN}-\exp\frac{2\pi e^{i\lambda}z_k[x']}{TN}}+ \tag{13}$$
$$+\frac{i\,e^{i\lambda}}{TN}\sum_{k=0}^{N-1}\varepsilon_k\cdot\int_{1}^{\infty}e^{-j\omega x'}\cdot\frac{\exp\frac{2\pi e^{i\lambda}z}{TN}\cdot dx'}{\exp\frac{2\pi e^{i\lambda}z}{TN}-\exp\frac{2\pi e^{i\lambda}z_k[x']}{TN}}.$$

The unknown vortex densities $\gamma_k(x')$ must now be evaluated from the conditions

$$\mathrm{Im}_i\{U_0\cdot W(z_m[x'])\} = -\,U_0\cdot g_m(x') \qquad \text{for } -1 < x' < +1,$$

m any integer. To the latter conditions we add the Kutta condition $W(z)=O(1)$ as $z\to z_m[1]$. Let us write the kernel in the formula for $W(z)$ in the following way:

$$\frac{i\,e^{i\lambda}}{TN}\,\frac{\exp\frac{2\pi e^{i\lambda}z}{TN}}{\exp\frac{2\pi e^{i\lambda}z}{TN}-\exp\frac{2\pi e^{i\lambda}\cdot z_k[x']}{TN}} = \frac{1}{2\pi i}\cdot\frac{1}{z_k[x']-z}+R\big(z-z_k[x']\big), \tag{14}$$

$R\big(z-z_k[x']\big)$ being holomorphic in a full neighborhood of $z=z_k[x']$. Then by [*17*, p. 84 *seq.*] we see that $W(z)=O(1)$ since $z\to z_k[1]$ is possible only if the distribution densities of the "generalized integrals of the Cauchy type" in (13) are continuous in $x'=1$; *i.e.*, if

$$\gamma_k(1) = \varepsilon_k\cdot e^{-j\omega} \qquad \text{for } k=0,1,\ldots,N-1. \tag{15}$$

The boundary conditions for the normal velocities on the line profiles result in a system of singular integral equations for the $\gamma_k(x')$ with CAUCHY's principal values in $-1<x'<+1$. We have

$$g_l(x) = -\frac{1}{4TN}\cdot\sum_{k=0}^{N-1}\int_{-1}^{+1}\gamma_k(x')\cdot K(x,x';l-k,\lambda)\,dx' - \tag{16}$$
$$-\frac{1}{4TN}\cdot\sum_{k=0}^{N-1}\varepsilon_k\cdot\int_{+1}^{\infty}e^{-j\omega x'}\cdot K(x,x';l-k,\lambda)\,dx'$$
$$\text{for} \quad l=0,1,\ldots,N-1$$

with

$$K(x,x';l-k,\lambda)$$
$$=\frac{\cos\lambda\cdot\exp\left[\frac{2\pi}{TN}\cdot(x-x')\cos\lambda\right]-\cos\left\{\lambda+\frac{2\pi}{TN}\cdot[(x-x')\sin\lambda+T(l-k)]\right\}}{\sinh^2\left[\frac{\pi}{TN}\cdot(x-x')\cos\lambda\right]+\sin^2\left\{\frac{\pi}{TN}\cdot[(x-x')\sin\lambda+T(l-k)]\right\}}.$$

In order to abbreviate let us write

$$h_{l,k}(x;\lambda) = -\frac{1}{4TN}\cdot\int_{1}^{\infty}e^{-j\omega x'}\cdot K(x,x';l-k,\lambda)\,dx' \tag{17}$$

for $k, l=0, 1, \ldots, N-1$ and $-1 \leqq x \leqq +1$. Then we see that $h_{l,k}(x;\lambda)$ is dependent only on the class $l-k \equiv r(N)$. Thus we may write $h_{l,k}(x;\lambda)=h_r(x;\lambda)$. Considering the ε_k as known, we can transform the above system of integral equations into the following:

$$
\begin{aligned}
g_l(x) - \sum_{k=0}^{N-1} \varepsilon_k \cdot h_{l,k}(x;\lambda) &= q_l(x) = q\left(z_l[x]\right) \\
&= -\frac{1}{4TN} \cdot \sum_{k=0}^{N-1} \int_{-1}^{+1} \gamma_k(x') \cdot K(x, x'; l-k, \lambda)\, dx'
\end{aligned}
\tag{18}
$$

for $l=0, 1, \ldots, N-1$. This system of integral equations is also obtained from the problem of a steady flow passing through a staggered cascade of line profiles which slightly deflects the basic flow parallel to the x-axis, if the normal velocities $q_l(x')$ be prescribed on the profiles $z_0=c \cdot z_m[x']$ and without any trailing vortices in the wakes $z_0=c \cdot z_m[x']$, $x' \geqq 1$. Thus we have reduced the original unsteady problem to a steady one. After evaluating $\gamma_k(x')$, which must remain bounded according to equation (15) as $x' \to +1$, we still have to evaluate ε_k from the system of linear equations that results from equation (12).

From the kernel $K(x, x'; l-k, \lambda)$ we split off the singular part for $l=k$ and $x=x'$, and according to [*17*, p. 74] we then deduce that as $x \to +1$,

$$
h_{l,k}(x;\lambda) = \begin{cases} O\left(|\log|x-1||\right) & \text{for } k=l, \\ O(1) & \text{for } k \neq l. \end{cases}
\tag{19}
$$

From it results that as $x \to +1$,

$$
q_l(x) = O\left(|\log|x-1||\right) \quad \text{for } l=0, 1, \ldots, N-1,
\tag{20}
$$

if we assume $g_l(x)$ to be Hoelder-continuous for $-1 \leqq x \leqq +1$. The reduced and steady velocity field $U_0 \cdot W^*(z)$ we require is then set up as follows:

$$
W^*(z) = \frac{i e^{i\lambda}}{TN} \sum_{k=0}^{N-1} \int_{-1}^{+1} \gamma_k(x') \frac{\exp \dfrac{2\pi e^{i\lambda} z}{TN} \cdot dx'}{\exp \dfrac{2\pi e^{i\lambda} z}{TN} - \exp \dfrac{2\pi e^{i\lambda} z_k[x']}{TN}}.
\tag{21}
$$

By equation (15) we require the boundedness of $\gamma_k(1)$. After the splitting of the kernel according to equation (14) and according to [*17*], $W^*(z)$ as $z \to z_l[1]$ must behave like $O\left(|\log|z-z_l[1]||\right)$ for $l=0, 1, \ldots, N-1$. Once more our reduced problem may be precisely formulated as a function-theoretic one:

We require a complex velocity field $W^*(z)$ that is holomorphic in the z-plane with the exception of the slits $z=z_m[x']=x'+iT \cdot m \cdot e^{i\lambda}$ for $-1 \leqq x' \leqq +1$ and m any integer. This field shall satisfy the following conditions:

1. $W^*(z+iT \cdot N \cdot e^{-i\lambda}) = W^*(z)$ *for all* z
2. $W^*(z) = o(1)$ *as* $\operatorname{Re} z = x \to -\infty$
   $ = O(1)$ *as* $\operatorname{Re} z = x \to +\infty$
3. $\operatorname{Im}_i W^*(z) = -q_l(x)$ *for* $z=z_l[x] \pm i \cdot 0$ *in* $-1 < x < +1$, $l=0, 1, \ldots, N-1$
4. $W^*(z) = O\left(|\log|z-z_l[1]||\right)$ *as* $z \to z_l[1]$
5. $W^*(z) = O\left(|z-z_l[-1]|^{-\alpha_l}\right)$ *as* $z \to z_l[-1]$ *with* $0 \leqq \alpha_l < 1$.

3. Conformal Mapping

It is very unlikely that the problem defined in the last section can be solved directly in the z-plane because the system of integral equations (18) for the vortex densities $\gamma_k(x')$ is very complicated. We shall try to approach it in another way. The z-plane will be conformally mapped onto a ζ-plane in such a way that the staggered cascade of line profiles is transformed into a horizontal one. For the velocity field of a flat plate cascade in a fluid steadily passing through it, DURAND's book gives some formulae [*18*, p. 91] which help to obtain the required mapping function $z=f(\zeta)$. In the case of the vertical cascade we were also able to state explicitly the inverse function $\zeta=g(z)$ [*19*], but this will not be possible here.

From v. KÁRMÁN & BURGERS in [*18*] the formula for the conformal mapping of the exterior of the unit circle in the $\alpha=\alpha_1+i\cdot\alpha_2$-plane on to the exterior of the cascade of line profiles with the spacing T, the chord length 2 and the angle of stagger λ runs as follows:

$$z(\alpha)=\frac{T}{2\pi}\left[e^{-i\lambda}\log\frac{1/\varkappa+\alpha}{1/\varkappa-\alpha}+e^{i\lambda}\cdot\log\frac{\alpha+\varkappa}{\alpha-\varkappa}\right] \tag{22}$$

with $0<\varkappa<1$. There is a relation between T, $\varkappa$ and λ [*18*]:

$$\begin{aligned}\frac{\pi}{T}&=\cos\lambda\cdot\log\left[(1-\varkappa^2)^{-1}\cdot\{2\varkappa\cdot\cos\lambda+(1+\varkappa^4+2\varkappa^2\cos 2\lambda)^{\frac12}\}\right]+\\&\quad+\sin\lambda\cdot\tan^{-1}\left[2\varkappa\cdot\sin\lambda\cdot(1+\varkappa^4+2\varkappa^2\cdot\cos 2\lambda)^{-\frac12}\right].\end{aligned} \tag{23}$$

This equation cannot be solved explicitly for $\varkappa$ in the case of general λ. In all practical cases this equation requires numerical calculation which takes into eccount the restriction that $0<\varkappa<1$. A list of some numerical values for $\varkappa$, T, and λ is found in a paper by N. SCHOLZ [*20*].

Now we shall map the exterior of the unit circle in the α-plane onto the exterior of a cascade of line profiles in the $\zeta=\xi+i\cdot\eta$-plane with the spacing τ, the chord length 2, and the angle of stagger 90° by means of the function

$$\zeta(\alpha)=\frac{\tau\cdot i}{2\pi}\left[-\log\frac{1/\varkappa+\alpha}{1/\varkappa-\alpha}+\log\frac{\alpha+\varkappa}{\alpha-\varkappa}\right]. \tag{24}$$

The spacing τ must be evaluated from a relation similar to equation (23), namely,

$$\tau=\frac{\pi}{2\cdot\tan^{-1}\varkappa}>2, \tag{25}$$

where $\varkappa$ is known from (23).

The functions $z=z(\alpha)$ in (22) and $\zeta=\zeta(\alpha)$ in (24) effect in parametric form the conformal mapping of the exterior of a staggered cascade of line profiles onto a horizontal one. After eliminating α, we may write

$$z=f(\zeta)=-\frac{T\cdot\cos\lambda}{\pi}\cdot\left[\log\frac{\cos\frac{\pi\zeta}{\tau}+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\zeta}{\tau}\right)^{\frac12}}{\cos\frac{\pi}{\tau}}-\pi\cdot\tan\lambda\cdot\frac{\zeta}{\tau}\right]. \tag{26}$$

The square root here is to be determined in such a way that it is positive for real $\zeta=\xi$ on the upper side of the slit $-1\leqq\xi\leqq+1$, where the principal branch

for the logarithm is chosen. For further investigations we need the derivative $f'(\zeta)$:

$$\frac{dz}{d\zeta}=f'(\zeta)=\frac{T\cos\lambda}{\tau}\cdot\left[\sin\frac{\pi\zeta}{\tau}\cdot\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\zeta}{\tau}\right)^{-\frac{1}{2}}+\tan\lambda\right]. \tag{27}$$

The image points of the leading and trailing edges of the profiles in the z-plane may be obtained from $dz/d\zeta=0$ as

$$\begin{aligned}\zeta_L&=-\zeta_0+m\cdot\tau \qquad \text{at the upper bank,}\\ \zeta_T&=\zeta_0+m\cdot\tau \qquad \text{at the lower bank}\end{aligned} \tag{28}$$

with $\zeta_0=\frac{\tau}{\pi}\cdot\sin^{-1}\left(\sin\lambda\cdot\sin\frac{\pi}{\tau}\right)$ and m any integer.

Besides this it can easily be proved that the following points correspond to each other:

$$\zeta=1 \quad \text{and} \quad z=\frac{T}{\tau}\cdot\sin\lambda \quad \text{(upper bank)},$$

$$\zeta=-1 \quad \text{and} \quad z=-\frac{T}{\tau}\cdot\sin\lambda \quad \text{(lower bank)}.$$

In the special case $\lambda=\pi/2$ equation (26) reduces to $z=\zeta$ since $\tau=T$ then. If $T\to\infty$, $\varkappa\to 0$ by (23), and $\tau\to\infty$ by (25). It is obvious that in this limiting case the mapping functions (22) and (24) will tend to

$$z_\infty(\alpha)=\frac{1}{2}\left(e^{-i\lambda}\alpha+\frac{1}{e^{-i\lambda}\alpha}\right), \tag{29a}$$

$$\zeta_\infty(\alpha)=-\frac{1}{2}\left(\alpha-\frac{1}{\alpha}\right), \tag{29b}$$

so that equation (26) becomes

$$z_\infty(\zeta)=\sin\lambda\cdot\zeta+i\cdot\cos\lambda\cdot(\zeta^2-1)^{\frac{1}{2}}, \tag{29c}$$

where the square root is taken positive for $\zeta=\xi>1$ and the branch cut is $-1\leqq\zeta\leqq+1$.

4. Solution of the Transferred Boundary Value Problem

We now transfer our problem into the ζ-plane in order to determine the complex flow field $W^*(z)$. If $\Phi=F[z(\zeta)]$ denotes the complex velocity potential of the flow field to be ascertained in the ζ-plane, the following terms express the well known relations between the velocity fields in the ζ-plane and those in the z-plane:

$$u(\xi,\eta)-i\cdot v(\xi,\eta)=\Omega(\zeta)=\frac{dF}{dz}\cdot\frac{dz}{d\zeta}=W^*(z)\cdot f'(\zeta). \tag{30}$$

Since the field $W^*(z)$ is required to have the period $iTNe^{-i\lambda}$, the field $\Omega(\zeta)$ must be characterized by the period $\tau\cdot N$. From equation (27) we easily infer that $dz/d\zeta\to iTe^{-i\lambda}$ as $|\mathrm{Im}\,\zeta|=|\eta|\to\infty$. The conditions $W^*(z)=o(1)$ as $\mathrm{Re}\,z=x\to-\infty$ and $=O(1)$ as $\mathrm{Re}\,z=x\to+\infty$ then change into $\Omega(\zeta)=o(1)$ as $\mathrm{Im}\,\zeta=\to+\infty$ and $=O(1)$ as $\mathrm{Im}\,\zeta=\eta=\to-\infty$. In transferring the boundary conditions $\mathrm{Im}\,W^*(z)=-q_l(x)$ for $z=z_l[x]\pm i\cdot 0$ with $-1<x<+1$ and $l=0,1,\ldots,N-1$, we have to take care that these conditions be satisfied on

both sides of the line profiles $z=z_l[x]$. On the staggered cascade the prescribed boundary values $q_l(x)$ are equal on either side of the segments $z=z_l[x]$, but in general they are no longer so after conformal mapping onto the ζ-plane. The boundary values on the upper side of the slits $\tau\cdot l-1\leqq\xi\leqq\tau\cdot l+1$, $\eta=0$ must be distinguished strictly from those on the lower side of these cuts. On application of (30), we can state the following boundary values: $\operatorname{Im}\{\Omega(\zeta)\cdot d\zeta/dz\}=-q_l(x)=-q(z_l[x])$ for $z_l[x]=f(\xi)$ in $-1<x<+1$ or $\tau\cdot l-1<\xi<\tau\cdot l+1$, respectively, with $l=0, 1, \ldots, N-1$. From the above boundary values we obtain, after some transformations, the following ones for $\Omega(\zeta)=u(\xi,\eta)-i\cdot v(\xi,\eta)$:

$$(31)\qquad \begin{aligned} v(\xi,+0)&=q[f(\xi+i\cdot 0)]\cdot\frac{T\cdot\cos\lambda}{\tau}\cdot\left\{\frac{\sin\pi\left(\frac{\xi}{\tau}-l\right)}{{}_{+}\!\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\,\xi}{\tau}\right)^{\frac{1}{2}}}+\tan\lambda\right\}=r_l^+(\xi),\\ v(\xi,-0)&=q[f(\xi-i\cdot 0)]\cdot\frac{T\cdot\cos\lambda}{\tau}\cdot\left\{\frac{-\sin\pi\left(\frac{\xi}{\tau}-l\right)}{{}_{+}\!\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\,\xi}{\tau}\right)^{\frac{1}{2}}}+\tan\lambda\right\}=r_l^-(\xi)\end{aligned}$$

for $\tau\cdot l-1<\xi<\tau\cdot l+1$; $l=0, 1, \ldots, N-1$. The square roots ${}_{+}(\ldots)^{\frac{1}{2}}$ must be taken as positive.

We now study the behavior of the newly prescribed boundary values $r_l^{\pm}(\xi)$ on the lines $\tau\cdot l-1<\xi<\tau\cdot l+1$. If $H(z)$ is any function in the z-plane, which need not be analytic nor defined in a region except on arcs, and for which the following estimates are valid (m an integer), we have

$$(32\text{a})\qquad H(z)=\begin{cases} O\left(|z-z_m[-1]|\right) & \text{for } z\to z_m[-1]\\ O\left(|z-z_m[1]|\right) & \text{for } z\to z_m[+1]\\ O\left(\left|z-z_m\left[\frac{T}{\tau}\sin\lambda\right]\right|\right) & \text{for } z\to z_m\left[\frac{T}{\tau}\sin\lambda\right]+i\cdot 0\\ O\left(\left|z-z_m\left[-\frac{T}{\tau}\sin\lambda\right]\right|\right) & \text{for } z\to z_m\left[-\frac{T}{\tau}\sin\lambda\right]-i\cdot 0;\end{cases}$$

then for the function $H[z(\zeta)]$ transplanted into the ζ-plane the following estimates will hold in the image points:

$$(32\text{b})\qquad H[z(\zeta)]=\begin{cases} O\left(|\zeta-\tau\cdot m+\zeta_0|^2\right) & \text{for } \zeta\to\tau\cdot m-\zeta_0+i\cdot 0\\ O\left(|\zeta-\tau\cdot m-\zeta_0|^2\right) & \text{for } \zeta\to\tau\cdot m+\zeta_0-i\cdot 0\\ O\left(|\zeta-\tau\cdot m-1|^{\frac{1}{2}}\right) & \text{for } \zeta\to\tau\cdot m+1\\ O\left(|\zeta-\tau\cdot m+1|^{\frac{1}{2}}\right) & \text{for } \zeta\to\tau\cdot m-1.\end{cases}$$

Since $q(z_l[x])=O(1)$ for $x\to-1$ and $=O\left(|\log|x-1||\right)$ as $x\to+1$, $l=0, 1, \ldots, N-1$, the new boundary values have the following properties:

$$(33)\qquad \begin{aligned} r_l^+(\xi)&=\begin{cases} O\left(|\xi-\tau\cdot l+\zeta_0|\right) & \text{for } \xi\to l\cdot\tau-\zeta_0\\ O\left(|\xi-\tau\cdot l\pm 1|^{-\frac{1}{2}}\right) & \text{for } \xi\to l\cdot\tau\mp 1,\end{cases}\\ r_l^-(\xi)&=\begin{cases} O\left(|\xi-l\cdot\tau-\zeta_0|\cdot|\log|\xi-l\cdot\tau-\zeta_0||\right) & \text{for } \xi\to l\cdot\tau+\zeta_0\\ O\left(|\xi-l\cdot\tau\pm 1|^{-\frac{1}{2}}\right) & \text{for } \xi\to l\cdot\tau\mp 1.\end{cases}\end{aligned}$$

From the behavior of $W^*(z)$ at the points $z=z_m[+1]$, m an integer, and from the fact that $W^*(z)=O(1)$ as $z\to z_m[x]$ for fixed x when $-1<x<+1$, and from taking into consideration equations (30), (27), and (32), we infer the appropriate behavior of the transformed field $\Omega(\zeta)$ as follows:

$$\Omega(\zeta)=\begin{cases} O\left(|\zeta-\tau\cdot m-\zeta_0|^{-2\alpha_m+1}\right) & \text{for } \zeta\to\tau\cdot m-\zeta_0+i\cdot 0 \\ O\left(|\zeta-\tau\cdot m-\zeta_0|\cdot|\log|\zeta-\tau\cdot m-\zeta_0||\right) & \text{for } \zeta\to\tau\cdot m+\zeta_0-i\cdot 0 \\ O\left(|\zeta-\tau\cdot m\pm 1|^{-\frac{1}{2}}\right) & \text{for } \zeta\to\tau\cdot m\mp 1. \end{cases}$$

Since $0\leqq\alpha_m<1$, we write $-2\alpha_m+1=-\mu_m$ with $0\leqq\mu_m<1$, and we know that on the slits $\tau\cdot m-1\leqq\xi\leqq\tau\cdot m+1$, $\eta=0$, $\Omega(\zeta)$ has at most integrable singularities. Since in general the prescribed boundary values $r_l^+(\xi)$ and $r_l^-(\xi)$ on both sides of the slits are not equal to each other, we construct the required flow field $\Omega(\zeta)$ from two others, namely $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$, in the following way: We define new prescribed boundary values $r_{l1}(\xi)$ and $r_{l2}(\xi)$ by

$$\begin{aligned} r_{l1}(\xi)&=\tfrac{1}{2}[r_l^+(\xi)+r_l^-(\xi)] \\ r_{l2}(\xi)&=\tfrac{1}{2}[r_l^+(\xi)-r_l^-(\xi)] \end{aligned} \qquad l=0,1,\ldots,N-1 \tag{34}$$

on the upper sides and by $r_{l1}(\xi)$ and $-r_{l2}(\xi)$ on the lower sides of the slits $\tau\cdot l-1\leqq\xi\leqq\tau\cdot l+1$, $\eta=0$, respectively. Let the field $\Omega_1(\zeta)$ induced by vortex distributions have the boundary values $r_{l1}(\xi)$, and let the field $\Omega_2(\zeta)$ induced by source distributions have the boundary values $\pm r_{l2}(\xi)$, respectively. Now we specify our problem for the two functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$:

We require two complex velocity fields, $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$, that are holomorphic and periodic with the period $\tau\cdot N$ in the $\zeta=\xi+i\cdot\eta$-plane cut along the segments $\tau\cdot m-1\leqq\xi\leqq\tau\cdot m+1$, $\eta=0$. As $|\eta|\to\infty$ these two fields shall remain bounded, but $\Omega_1(\zeta)+\Omega_2(\zeta)$ shall tend to zero as $\eta\to+\infty$. When ζ approaches the real ζ-axis, the following conditions shall hold:

$$\lim_{\eta\to+0}\operatorname{Im}\Omega_1(\zeta)=\lim_{\eta\to-0}\operatorname{Im}\Omega_1(\zeta)=-r_{l1}(\xi),$$
$$\lim_{\eta\to+0}\operatorname{Im}\Omega_2(\zeta)=-\lim_{\eta\to-0}\operatorname{Im}\Omega_2(\zeta)=-r_{l2}(\xi)$$

for $l=0,1,\ldots,N-1$. In the limiting points of the segments, $\zeta=\tau\cdot m\pm 1$, and in $\zeta=\tau\cdot m\pm\zeta_0$, m any integer, $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ shall have at most integrable singularities. For $\zeta\to\tau\cdot m+\zeta_0$ on the lower side it is required further (Kutta condition) that

$$[\Omega_1(\zeta)+\Omega_2(\zeta)]\cdot\frac{d\zeta}{dz}=O\left(|\log|\zeta-\tau\cdot m-\zeta_0||\right).$$

It is easily understood that a homogeneous flow field parallel to the real ζ-axis can be superposed without any changes in the boundary conditions and the O-conditions on the slits. As $|\eta|\to\infty$, the field $\Omega_1(\zeta)+\Omega_2(\zeta)$ plus this parallel field remains bounded. On the other hand, by superposing such a field on the field $\Omega_1(\zeta)+\Omega_2(\zeta)$, we can make the total field vanish as $\eta\to+\infty$, provided $\lim[\Omega_1(\zeta)+\Omega_2(\zeta)]$ be real and $\neq 0$ as $\eta\to+\infty$. In the following it is sufficient therefore to assume that $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ are bounded as $|\eta|\to\infty$ and that $\Omega_1(\zeta)+\Omega_2(\zeta)$ tend to a real limit as $\eta\to+\infty$.

We start with the evaluation of the function $\Omega_2(\zeta)$, since this can be done more easily. At the points $\zeta=\omega_k+n\cdot N\tau$, n an arbitrary integer and with fixed ω_k in $\tau\cdot k-1<\omega_k<\tau\cdot k+1$, $k=0,1,\ldots,N-1$, we place sources of equal strengths $Q_k(\omega_k)\,d\omega_k$. Then the field of such a series of sources is defined by

$$d\Omega_2=Q_k(\omega_k)\,d\omega_k\cdot\frac{1}{2\tau N}\cdot\operatorname{ctn}\frac{\pi}{\tau\cdot N}(\zeta-\omega_k).$$

For the total source field that results from the integration and summation over $\tau\cdot k-1<\omega_k<\tau\cdot k+1$ and over k from 0 to $N-1$, respectively, we obtain the term

$$\Omega_2(\zeta)=\frac{1}{2\tau\cdot N}\sum_{k=0}^{N-1}\int_{\tau\cdot k-1}^{\tau\cdot k+1}Q_k(\omega_k)\cdot\operatorname{ctn}\frac{\pi}{N\tau}(\zeta-\omega_k)\,d\omega_k.\tag{35}$$

The still unknown source strengths $Q_k(\omega_k)$ must be evaluated from the boundary conditions. In a full neighborhood of $\zeta=\omega_k$ the kernel can be written in the form

$$\frac{1}{2\tau N}\cdot\operatorname{ctn}\frac{\pi}{\tau N}(\zeta-\omega_k)=-\frac{i}{2\pi i(\omega_k-\zeta)}+P(\zeta-\omega_k).$$

In this neighborhood $P(\zeta-\omega_k)$ is holomorphic. On consideration of this separation and the Plemelj formulae [*17*, p. 43], we obtain

$$\Omega_2(\xi+i\cdot 0)-\Omega_2(\xi-i\cdot 0)=-i\cdot Q_l(\xi)\tag{36a}$$

for $\tau\cdot l-1<\xi<\tau\cdot l+1$ and $l=0,1,\ldots,N-1$. Since

$$\operatorname{ctn}\frac{\pi}{\tau N}(\bar{\zeta}-\omega_k)=\overline{\operatorname{ctn}\frac{\pi}{\tau N}(\zeta-\omega_k)},$$

it follows that $\Omega_2(\bar{\zeta})=\overline{\Omega_2(\zeta)}$. We may then conclude that

$$\Omega_2(\xi+i\cdot 0)-\overline{\Omega_2(\xi+i\cdot 0)}=-i\cdot Q_l(\xi).\tag{36b}$$

On the other hand, this is equal to $2i\cdot\operatorname{Im}\cdot\Omega_2(\xi+i\cdot 0)=-2i\cdot r_{l2}(\xi)$ due to the boundary conditions. By comparison it follows that $Q_l(\omega_l)=2\cdot r_{l2}(\omega_l)$ in $\tau\cdot l-1<\omega_l<\tau\cdot l+1$. Inserting this result into equation (35), we obtain

$$\Omega_2(\zeta)=\frac{1}{\tau N}\sum_{k=0}^{N-1}\int_{\omega_k=\tau k-1}^{\tau k+1}r_{k2}(\omega_k)\cdot\operatorname{ctn}\frac{\pi}{\tau\cdot N}(\zeta-\omega_k)\cdot d\omega_k.\tag{37}$$

Owing to the behavior of $r_k^{\pm}(\omega_k)$ and consequently to that of $r_{k2}(\omega_k)$ at the limiting points $\omega_k=\tau\cdot k\pm 1$, the growth properties of $\Omega_2(\zeta)$ in $\zeta=\tau\cdot m\pm 1$, m any integer, in view of [*17*, p. 74] are as follows: $\Omega_2(\zeta)=O\left(|\zeta-\tau\cdot m\mp 1|^{-\frac{1}{2}}\right)$. Since the density functions $r_{k2}(\omega_k)$ are otherwise Hoelder-continuous, $\Omega_2(\zeta)$ remains bounded when ζ approaches the inner points of the slit sides. Since $\lim\operatorname{ctn}\frac{\pi}{\tau N}(\zeta-\omega_k)=\mp i$ as $\operatorname{Im}\zeta=\eta\to\pm\infty$, we have

$$\lim_{\eta\to\pm\infty}\Omega_2(\zeta)=\mp\frac{i}{N\tau}\cdot\sum_{k=0}^{N-1}\int_{\omega_k=\tau\cdot k-1}^{\tau\cdot k+1}r_{k2}(\omega_k)\,d\omega_k.\tag{38}$$

Herein we substitute $\omega_k = \omega_k(z_k[x'])$ for $\tau \cdot k - 1 < \omega_k < \tau \cdot k + 1$ and $-1 < x' < +1$, and after some transformations we may rewrite the last term as

$$\lim_{\eta \to \pm\infty} \Omega_2(\zeta) = \mp \frac{i}{2\tau N} \cdot \sum_{k=0}^{N-1} \int_{x'=-1}^{+1} [q(z_k[x'] + i \cdot 0) - q(z_k[x'] - i \cdot 0)]\, dx' = 0,$$

since the prescribed values $q_k(x') = q(z_k[x'])$ on both sides of the line profiles $z = z_k[x']$ with $-1 \leqq x' \leqq +1$ and $k = 0, 1, \ldots, N-1$ are the same. The total source strengths in the ζ-plane vanish, resulting from the fact that they vanish for the field $W^*(z)$ in the z-plane and do not change their values after conformal mapping. Using (34) and (31), we insert the terms for $r_{k2}(\omega_k)$, substitute $\omega_k = \tau \cdot k + \sigma$ where $-1 < \sigma < +1$, and obtain the following terms for $\Omega_2(\zeta)$:

$$\begin{aligned} \Omega_2(\zeta) = \frac{T \cdot \cos\lambda}{2N\tau^2} \sum_{k=0}^{N-1} \int_{\sigma=-1}^{+1} \operatorname{ctn} \frac{\pi}{\tau N} (\zeta - \sigma - \tau \cdot k) \times \\ \times \left[\left(q_k^+(\sigma) - q_k^-(\sigma)\right) \cdot \tan\lambda + \left(q_k^+(\sigma) + q_k^-(\sigma)\right) \frac{\sin \frac{\pi\sigma}{\tau}}{\left(\sin^2 \frac{\pi}{\tau} - \sin^2 \frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}} \right] d\sigma. \end{aligned} \tag{39}$$

In order to abbreviate we have put $q_k^{\pm}(\sigma) = q[f(\sigma + \tau \cdot k \pm i \cdot 0)]$, $k = 0, 1, \ldots, N-1$. Hence the complex velocity field $\Omega_2(\zeta)$ is determined.

Now we shall determine the general solution for the field $\Omega_1(\zeta)$. As stated in [*19*], we may start with vortex distributions on the slits, and, applying the NICKEL formulae [*21*], we may evaluate the vortex densities from a system of singular integral equations. In the present case, however, it serves our purpose better to reformulate the problem as a boundary-value problem of the Hilbert type and apply to it those methods given in [*17*] that were so useful in reaching our goal in [*19*]. For a field $\Omega_1(\zeta)$ induced by vortex distributions along the slits $\tau \cdot m - 1 \leqq \xi \leqq \tau \cdot m + 1$, $\eta = 0$, m any integer, we know that

$$\Omega_1(\zeta) = -\overline{\Omega_1(\bar{\zeta})}. \tag{40}$$

Due to the boundary conditions for $\Omega_1(\zeta)$ we can deduce for $\tau \cdot l - 1 < \xi < \tau \cdot l + 1$, $l = 0, 1, \ldots, N-1$

$$\begin{aligned} \Omega_1(\xi + i \cdot 0) &= -\overline{\Omega_1(\xi - i \cdot 0)}, \\ \text{or} \quad &= -\Omega_1(\xi - i \cdot 0) + 2i \cdot \operatorname{Im} \Omega_1(\xi - i \cdot 0), \\ \text{or} \quad &= -\Omega_1(\xi - i \cdot 0) - 2i \cdot r_{l1}(\xi); \end{aligned} \tag{41 a}$$

between the slits, *i.e.*, $\tau \cdot m + 1 < \xi < \tau(m+1) - 1$, m any integer, $\Omega_1(\zeta)$ is holomorphic, so that

$$\Omega_1(\xi + i \cdot 0) = \Omega_1(\xi - i \cdot 0). \tag{41 b}$$

Equations (41 a, b) express an inhomogenous, linear Hilbert boundary-value problem for the function $\Omega_1(\zeta)$. Some additional conditions are

$$\Omega_1(\zeta) = \Omega_1(\zeta + \tau \cdot N) \qquad \text{for all } \zeta, \tag{41 c}$$

$$\Omega_1(\zeta) = \begin{cases} O\left(|\zeta - \tau \cdot m \mp 1|^{-\mu_m}\right) & \text{for } \zeta \to \tau \cdot m \pm 1, \quad 0 \leqq \mu_m < 1, \quad \text{or} \\ O\left(|\zeta - \tau \cdot m \pm \zeta_0|^{-k_m}\right) & \text{for } \zeta \to \tau \cdot m - \zeta_0 \end{cases} \tag{41 d}$$

on the upper side and for $\zeta \to \tau \cdot m + \zeta_0$ on the lower side of the slit with $0 \leqq k_m < 1$,

$$\Omega_1(\zeta) = O(1) \quad \text{as} \quad |\eta| \to \infty. \tag{41 e}$$

As is known, the general inhomogeneous solution of such a linear Hilbert boundary-value problem is obtained by superposition of any arbitrary inhomogeneous solution and all the homogeneous solutions [17]. First we determine the totality of homogeneous solutions, and for the time being we distinguish between the cases of even and odd N.

a) N is an even number. It can easily be verified that the function

$$\Omega_1^0(\zeta) = \frac{T \cdot \cos\lambda}{2N \cdot i \cdot \tau^2} \left[\sin\frac{\pi}{\tau}(\zeta+1) \cdot \sin\frac{\pi}{\tau}(\zeta-1)\right]^{-\frac{1}{2}} \tag{42}$$

satisfies all the conditions made in (41) concerning $r_{11}(\xi)=0$ and that $\mu_m = \frac{1}{2}$, $k_m = 0$. Besides this we obtain

$$\Omega_1^0(\zeta) = O\left(\exp\left(-\frac{\pi}{\tau}|\eta|\right)\right) \quad \text{as} \quad |\eta| \to \infty. \tag{43}$$

The square root in (42) is to be determined so that

$$\left[\sin\frac{\pi}{\tau}(\zeta+1) \cdot \sin\frac{\pi}{\tau}(\zeta-1)\right]^{\frac{1}{2}} = i \cdot {}_{+}\!\left[\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\zeta}{\tau}\right]^{\frac{1}{2}}$$

with the second root being positive for $\zeta = \xi + i \cdot 0$ and $-1 < \xi < +1$. The slits $\tau \cdot m - 1 \leqq \xi \leqq \tau \cdot m + 1$, $\eta = 0$, m being any integer, are the branch cuts of $\Omega_1^0(\zeta)$.

Assuming $X(\zeta)$ to be any other solution of the homogeneous Hilbert problem with the properties of equations (41), we write the quotient

$$H(\zeta) = \frac{X(\zeta)}{\Omega_1^0(\zeta)}. \tag{44}$$

Concerning all real points $\zeta = \xi$ with the possible exception of $\zeta = \tau \cdot m \pm 1$ or $\zeta = \tau \cdot m \pm \zeta_0$, m any integer, this function $H(\zeta)$ has the property $H(\xi + i \cdot 0) = H(\xi - i \cdot 0)$; *i.e.*, it must be meromorphic and also periodic with the period $\tau \cdot N$ due to (41c). The conditions (41d) require of $H(\zeta)$ orders of infinity which are less than 1 in the above exceptional points. These, however, can only be removable singularities for a meromorphic function. Thus $H(\zeta)$ is known to be an entire function with the period $\tau \cdot N$ and can be expanded into a Fourier series which is convergent in the whole plane:

$$H(\zeta) = \sum_{\nu=-\infty}^{+\infty} a_\nu \cdot \exp\left(2\pi i \frac{\nu\zeta}{N\tau}\right).$$

Since $\Omega_1^0(\bar{\zeta}) = -\overline{\Omega_1^0(\zeta)}$ and $X(\bar{\zeta}) = -\overline{X(\zeta)}$, it follows that $H(\bar{\zeta}) = \overline{H(\zeta)}$. This results in a relation among the Fourier coefficients of $H(\zeta)$: $a_{-\nu} = \bar{a}_\nu$ for $\nu = 0, 1, 2, \ldots$. We required further that

$$X(\zeta) = H(\zeta) \cdot \Omega_1^0(\zeta) = O(1) \quad \text{as} \quad |\eta| \to \infty.$$

By means of equation (43) we obtain

$$e^{-\frac{\pi}{\tau}|\eta|} \cdot \left|\sum_{\nu=-\infty}^{\infty} a_\nu \cdot \exp\left(-2\pi \frac{\nu\eta}{\tau N}\right)\right| \leqq C \quad \text{as} \quad |\eta| \to \infty.$$

This, however, is possible only if $a_\nu = 0$ for $|\nu| \geqq N/2 + 1$.

We see that $H(\zeta)$ is reduced to a finite Fourier polynomial:

$$H(\zeta) = \sum_{\nu=-N/2}^{+N/2} a_\nu \cdot \exp\left(2\pi i \frac{\nu\zeta}{N\tau}\right) \quad \text{with } a_{-\nu} = \bar{a}_\nu$$

containing the free complex constants a_ν. Then the general solution of the homogeneous Hilbert boundary-value problem necessarily has the following form:

$$X(\zeta) = \frac{T\cos\lambda}{2N\cdot i\cdot\tau^2}\left[\sin\frac{\pi}{\tau}(\zeta+1)\cdot\sin\frac{\pi}{\tau}(\zeta-1)\right]^{-\frac{1}{2}}\times \tag{45}$$
$$\times\left\{c_0+\sum_{\nu=1}^{N/2} c_\nu\cdot\cos\left(2\pi\frac{\nu\,\zeta}{N\,\tau}\right)+d_\nu\cdot\sin\left(2\pi\frac{\nu\,\zeta}{N\,\tau}\right)\right\}$$

with $N+1$ arbitrary real constants. It is easily verified that all those properties mentioned under equations (41) actually hold for the functions $X(\zeta)$ with any real c_ν, d_ν, in particular with $\mu_m=\frac{1}{2}$, $k_m=0$ as here.

Now we construct an inhomogeneous solution. For this purpose we consider

$$\frac{\Omega_1^0(\xi+i\cdot 0)}{\Omega_1^0(\xi-i\cdot 0)} = \begin{cases} -1 & \text{for } \tau\cdot m-1<\xi<\tau\cdot m+1 \\ +1 & \text{for } \tau\cdot m+1<\xi<\tau(m+1)-1. \end{cases}$$

This we insert into equations (41a, b), respectively, make a little transformation, and then obtain

$$\frac{\Omega_1(\xi+i\cdot 0)}{\Omega_1^0(\xi+i\cdot 0)} = \frac{\Omega_1(\xi-i\cdot 0)}{\Omega_1^0(\xi-i\cdot 0)} + \begin{cases} \dfrac{-2i\cdot r_{l1}(\xi)}{\Omega_1^0(\xi+i\cdot 0)} & \text{in } \tau\cdot l-1<\xi<\tau\cdot l+1 \\ 0 & \text{in } \tau\cdot l+1<\xi<\tau(l+1)-1. \end{cases} \tag{46a}$$

Further, we deduce from results on page 210 that

$$\frac{\Omega_1(\zeta+\tau N)}{\Omega_1^0(\zeta+\tau N)} = \frac{\Omega_1(\zeta)}{\Omega_1^0(\zeta)} \quad \text{for all } \zeta. \tag{46b}$$

Due to the splitting of the ctn-kernel as described earlier and according to the Plemelj formulae *(loc. cit.)*, this inhomogenous Hilbert problem is solved by

$$\frac{\Omega_1(\zeta)}{\Omega_1^0(\zeta)} = \frac{i}{2N\tau}\cdot\sum_{k=0}^{N-1}\int_{\omega_k=\tau\cdot k-1}^{\tau\cdot k+1}\frac{-2i\cdot r_{k1}(\omega_k)}{\Omega_1^0(\omega_k+i\cdot 0)}\operatorname{ctn}\frac{\pi}{N\tau}(\zeta-\omega_k)\,d\omega_k.$$

Using (34) and (31), we substitute the terms for $r_{k1}(\omega_k)$ and consider

$$\Omega_1^0(\omega_k+i\cdot 0) = (-1)^{k+1}\cdot\frac{T\cdot\cos\lambda}{2N\tau^2}\cdot{}_+\left[\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\,\omega_k}{\tau}\right]^{-\frac{1}{2}} \tag{47}$$

with the positive square root; next we put $\omega_k=\sigma+\tau\cdot k$. Then we are able to state the general inhomogenous solution of the linear Hilbert boundary value problem with $N+1$ free real constants as follows:

$$\Omega_1(\zeta) = -\frac{T\cdot\cos\lambda}{2N\cdot i\cdot\tau^2}\left[\sin\frac{\pi}{\tau}(\zeta+1)\cdot\sin\frac{\pi}{\tau}(\zeta-1)\right]^{-\frac{1}{2}}\times$$
$$\times\sum_{k=0}^{N-1}(-1)^k\int_{\sigma=-1}^{+1}\operatorname{ctn}\frac{\pi}{\tau N}(\zeta-\sigma-\tau k)\times$$
$$\times\left[\left(q_k^+(\sigma)+q_k^-(\sigma)\right)\cdot\tan\lambda+\left(q_k^+(\sigma)-q_k^-(\sigma)\right)\frac{\sin\frac{\pi\sigma}{\tau}}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}\right]\times \tag{48}$$
$$\times\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}d\sigma+\frac{T\cdot\cos\lambda}{2N\cdot i\cdot\tau^2}\left[\sin\frac{\pi}{\tau}(\zeta+1)\cdot\sin\frac{\pi}{\tau}(\zeta-1)\right]^{-\frac{1}{2}}\times$$
$$\times\left\{c_0+\sum_{\nu=1}^{N/2}c_\nu\cdot\cos\left(2\pi\frac{\nu\,\zeta}{N\,\tau}\right)+d_\nu\cdot\sin\left(2\pi\frac{\nu\,\zeta}{N\,\tau}\right)\right\}.$$

As $\zeta \to \tau m \pm 1$ and $\zeta \to \tau \cdot m \pm \zeta_0$ this general solution has the required O-behavior with the special exponents $\mu_m = \frac{1}{2}$, $k_m = 0$, for the functions $q_k^+(\sigma)$ are Hoelder-continuous everywhere and the $q_k^-(\sigma)$ fail to be Hoelder-continuous at just one point apiece. These latter have logarithmic singularities at $\sigma = \zeta_0$, which correspond to the trailing edges $z_k[1]$ of the original cascade. In the formula above the $q_k^-(\sigma)$ are multiplied by the function $\tan\lambda\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}} - \sin\frac{\pi\sigma}{\tau}$, which vanishes at these points in first order.

Now the remaining conditions for the total field $\Omega(\zeta) = \Omega_1(\zeta) + \Omega_2(\zeta)$ must be satisfied. Let us begin with the requirement $\Omega(\zeta) \to 0$ as $\eta \to +\infty$. Now

$$\lim_{\eta \to +\infty} \Omega_1(\zeta) = -\frac{T \cdot \cos\lambda}{2N\tau^2}(c_{N/2} + i \cdot d_{N/2}). \tag{49}$$

We substitute the new field $\Omega_1(\zeta) + \Omega_1^1(\zeta)$ for the old field $\Omega_1(\zeta)$. Since $\lim \Omega_2(\zeta) = 0$ as $\eta \to +\infty$ and so as to avoid any change of the boundary conditions, we put

$$\Omega_1^1(\zeta) = \frac{T\cos\lambda}{2N\cdot\tau^2}\cdot c_{N/2} \quad \text{and} \quad d_{N/2} = 0. \tag{50}$$

The N real constants, $c_0, \ldots, c_{N/2}$ and $d_1, \ldots, d_{N/2-1}$, are still free; they are evaluated from the conditions

$$[\Omega_1(\zeta) + \Omega_1^1(\zeta) + \Omega_2(\zeta)]\cdot\frac{d\zeta}{dz} = O\left(|\log|\zeta - \tau\cdot m - \zeta_0||\right)$$

as $\zeta \to \tau\cdot m + \zeta_0 - i\cdot 0$, m being any integer. By the aid of the Plemelj formulae [*loc. cit.*] and (48), (39), and (27) we have

$$\begin{aligned}
&-i\cdot q_l[f(\xi - i\cdot 0)] + \left[(-1)^{l+1}\cdot\sin\frac{\pi\xi}{\tau} + \tan\lambda\cdot\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\xi}{\tau}\right)^{\frac{1}{2}}\right]^{-1}\times\\
&\times\frac{1}{2N\tau}\Bigg[{}_{+}\!\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\xi}{\tau}\right)^{\frac{1}{2}}\cdot\sum_{k=0}^{N-1}\int_{\sigma=-1}^{+1} r_{k2}(\sigma + \tau k)\operatorname{ctn}\frac{\pi}{\tau\cdot N}(\xi - \sigma - \tau\cdot k)\,d\sigma +\\
&+{}_{+}\!\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\xi}{\tau}\right)^{\frac{1}{2}}\cdot c_{N/2} - (-1)^l\cdot\sum_{k=0}^{N-1}(-1)^k\cdot\int_{\sigma=-1}^{+1} r_{k1}(\sigma + \tau k)\times\\
&\times{}_{+}\!\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}\operatorname{ctn}\frac{\pi}{\tau N}(\xi - \sigma - \tau\cdot k)\,d\sigma +\\
&+(-1)^l\left\{c_0 + \sum_{\nu=1}^{N/2-1} c_\nu\cdot\cos\left(2\pi\frac{\nu\xi}{N\tau}\right) + d_\nu\cdot\sin\left(2\pi\frac{\nu\xi}{N\tau}\right) + c_{N/2}\cdot\cos\frac{\pi\xi}{\tau}\right\}\Bigg]\\
&= O\left(|\log|\xi - \tau\cdot l - \zeta_0||\right)
\end{aligned} \tag{51}$$

for $\tau\cdot l - 1 < \xi < \tau\cdot l + 1$ and $\xi \to \tau\cdot l + \zeta_0$, $l = 0, 1, \ldots, N-1$. $r_{k1}(\sigma + \tau\cdot k)$ and $r_{k2}(\sigma + \tau\cdot k)$ are evaluated by means of (31) and (34). By equations (20) and (32), as $\xi \to \tau\cdot l + \zeta_0$, that is, as $z_l[x] \to z_l[1]$, the first term is $O\left(|\log|\xi - \tau\cdot l - \zeta_0||\right)$. The factor in brackets has a zero at $\xi = \tau\cdot l + \zeta_0$ by (27) and (28). Therefore the O-condition for the total term can be satisfied only if the N real coefficients $c_0, \ldots, c_{N/2}$ and $d_1, \ldots, d_{N/2-1}$ satisfy the system of linear equations that results from causing the term in the large bracket to vanish for $\xi = \tau\cdot l + \zeta_0$ with $l = 0, 1, \ldots, N-1$. This system of equations for the c_ν, d_ν can be transformed into one with separated unknowns by multiplying the l^{th} equation by $1/N\cdot\exp[-2\pi i n(\tau\cdot l + \zeta_0)/N\tau]$ and then summing over l.

This is done for $n=0, 1, \ldots, N/2$, and

$$c_0+\sum_{n=1}^{N/2-1} c_n \cdot \cos\left(2\pi \frac{n\zeta}{N\tau}\right)+d_n \cdot \sin\left(2\pi \frac{n\zeta}{N\tau}\right)$$

$$= -\sum_{k=0}^{N-1}(-1)^k \int_{\sigma=-1}^{+1} r_{k1}(\sigma+\tau\cdot k)\cdot \frac{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)\cdot \sin\frac{\pi}{N\tau}(-\tau\cdot k-\sigma)}\times$$

$$(52)\qquad \times\left[\sin\frac{\pi}{\tau N}\{(N-1)(\zeta-\tau k)+\sigma-N\cdot\zeta_0\}+\right.$$

$$+\sin\frac{\pi}{\tau N}\{(\zeta-\tau k)-(N+1)\sigma+N\cdot\zeta_0\}-$$

$$\left.-\cos\frac{\pi}{\tau}(\sigma-\zeta_0)\cdot\sin\frac{\pi}{\tau N}(\zeta-\tau\cdot k-\sigma)\right]\cdot d\sigma+$$

$$+\sum_{k=0}^{N-1}(-1)^k\int_{\sigma=-1}^{+1} r_{k2}(\sigma+\tau\cdot k)\cdot\sin\frac{\pi}{\tau}\cos\lambda\,\frac{\sin\left\{\frac{\pi}{N\tau}(N-1)(\zeta-\tau\cdot k-\sigma)\right\}}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)\cdot\sin\frac{\pi}{N}(\zeta-\tau\cdot k-\sigma)}\,d\sigma.$$

In the neighborhood of $\sigma=\zeta_0$ all the integrals occurring in the preceeding formulae are of the Cauchy principal value type. Noting that $d_{N/2}=0$, we substitute these last results into formula (48) for $\Omega_1(\zeta)$. Finally, the source field $\Omega_2(\zeta)$ of formula (39) and the homogenous parallel field $\Omega_1^1(\zeta)$ are added. Putting all this together, we obtain the field

$$\Omega(\zeta)=-\frac{1}{N i\tau}\left[\sin\frac{\pi}{\tau}(\zeta+1)\cdot\sin\frac{\pi}{\tau}(\zeta-1)\right]^{-\frac{1}{2}}\times$$

$$\times\sum_{k=0}^{N-1}\int_{\sigma=-1}^{+1}\left(r_{k1}(\sigma+\tau k)_{+}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}\times\right.$$

$$\times\left\{\frac{\sin\frac{\pi}{\tau N}[(N-1)\zeta+\sigma+\tau\cdot k-N\cdot\zeta_0]}{\sin\frac{\pi}{\tau N}(\zeta-\tau\cdot k-\sigma)}+\frac{\cos\frac{\pi\zeta}{\tau}}{\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi\zeta_0}{\tau}}\right\}-$$

$$(53)\qquad -r_{k2}(\sigma+\tau\cdot k)\cdot\sin\frac{\pi}{\tau}\cdot\cos\lambda\times$$

$$\left.\times\left\{\frac{\sin\frac{\pi}{\tau N}[(N-1)(\zeta-\sigma)+\tau\cdot k]}{\sin\frac{\pi}{\tau N}(\zeta-\sigma-\tau\cdot k)}+\frac{\cos\frac{\pi}{\tau}(\sigma-\zeta_0)\cdot\cos\frac{\pi\zeta}{\tau}}{\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi\zeta_0}{\tau}}\right\}\right)\frac{d\sigma}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}+$$

$$+\frac{1}{N\tau}\cdot\sum_{k=0}^{N-1}\int_{\sigma=-1}^{+1}\left(r_{k2}(\sigma+\tau k)\times\right.$$

$$\times\left\{\operatorname{ctn}\frac{\pi}{\tau N}(\zeta-\sigma-\tau\cdot k)\cdot\sin\frac{\pi}{\tau}(\sigma-\zeta_0)+\frac{\sin\frac{\pi}{\tau}\cdot\cos\lambda\cdot\cos\frac{\pi}{\tau}(\sigma-\zeta_0)}{\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi}{\tau}\zeta_0}\right\}-$$

$$\left.-r_{k1}(\sigma+\tau\cdot k)\frac{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}{\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi\zeta_0}{\tau}}\right)\frac{d\sigma}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}.$$

We evaluate the transformed prescribed values $r_{k1}(\sigma+\tau\cdot k)$ and $r_{k2}(\sigma+\tau\cdot k)$ from the $q[f(\sigma\pm i\cdot 0+\tau\cdot k)]$ by means of the formulas (31), and (34). The terms in the big parentheses following the integral sign have a zero term of first order in $\sigma=\zeta_0$ for all ζ, so that the zero term of the denominator which results from $\sin\frac{\pi}{\tau}(\sigma-\zeta_0)$ is removable. The above-mentioned formula (53) for $\Omega(\zeta)$ corresponds to formula (86) with $\lambda=0$ in [*19*]. In this paper the total circulation is assumed to vanish, so that our $\Omega(\zeta)$ must vanish as $|\eta|\to\infty$. In the present paper we see additional terms, *viz* those with the factor $\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi\zeta_0}{\tau}$ in the denominators.

b) N is an odd number. In the case when N is odd, we do not intend to state every single step of the investigation, since our approach would be the same as that already shown. There is no change at all in the source solution of the field $\Omega_2(\zeta)$. In order to find the fundamental solution of the homogenous Hilbert problem with equal boundary values on both sides of the segments, we again choose $\Omega_1^0(\zeta)$ by equation (42). Now we must take into account the fact that only this is correct: $\Omega_1^0(\zeta+\tau\cdot N)=-\Omega_1^0(\zeta)$. In order to obtain the special solution of the inhomogenous problem, instead of the ctn-kernel one must use $[\sin\pi(\zeta-\tau\cdot k-\sigma)/N\tau]^{-1}$ as the kernel. The general homogenous solution which remains bounded as $|\eta|\to\infty$ runs as follows:

$$
\begin{aligned}
X(\zeta) &= \frac{T\cdot\cos\lambda}{2N\cdot i\cdot\tau^2}\cdot\left[\sin\frac{\pi}{\tau}(\zeta+1)\cdot\sin\frac{\pi}{\tau}(\zeta-1)\right]^{-\frac{1}{2}}\times \\
&\quad\times\left\{\sum_{\nu=1}^{(N+1)/2} c'_\nu\cdot\cos\left[2\pi\left(\nu-\frac{1}{2}\right)\frac{\zeta}{N\tau}\right]+d'_\nu\cdot\sin\left[2\pi\left(\nu-\frac{1}{2}\right)\frac{\zeta}{N\tau}\right]\right\}
\end{aligned}
\tag{54}
$$

with $N+1$ arbitrary real constants c'_ν, d'_ν. These constants can be determined from the requirements that $\Omega(\zeta)\to 0$ as $\eta\to+\infty$ and $\Omega(\zeta)\cdot\frac{d\zeta}{dz}=O\left(|\log|\zeta-\tau\cdot l-\zeta_0||\right)$ as $\zeta\to\tau\cdot l+\zeta_0-i\cdot 0$, $l=0,1,\ldots,N-1$. Adding $\Omega_1(\zeta)$, $\Omega_2(\zeta)$, and $\Omega_1^1(\zeta)$, we again obtain $\Omega(\zeta)$ by (53). Thus we no longer need to distinguish between even and odd N.

The prescribed boundary values $q_k^\pm(\sigma)$ being known, we have solved our problem stated on page 208. Its solution is unique.

If we let T tend to infinity, we may conclude from (23) and (25) that $\varkappa\to 0$ and $\tau\to\infty$ in such a way that $\frac{T}{\tau}\to 1$. The mapping function (26) will then become that given by equation (29c), which may be solved for ζ by $\zeta=\sin\lambda\times z_\infty+\cos\lambda\cdot(1-z_\infty^2)^{\frac{1}{2}}$ with positive square root for $z_\infty=x_\infty+i\cdot 0$, $-1<x_\infty<+1$. Multiplying $\Omega(\zeta)$ in (53) by $d\zeta/dz$ and letting τ tend to infinity, we obtain the formula for the complex velocity field with a single line profile oscillating in an incompressible fluid flow, also neglecting the field induced by the trailing vortices as we did in the evaluation of $\Omega(\zeta)$; thus

$$
W_\infty^*(z_\infty)=-\frac{1}{\pi}\sqrt{\frac{1-z_\infty}{1+z_\infty}}\int_{x'=-1}^{+1}\sqrt{\frac{1+x'}{1-x'}}\,\frac{q(x')\,dx'}{x'-z_\infty}.
\tag{55}
$$

5. Determination of the Constants of the Trailing Vortices

Now we shall solve our original problem in the z-plane. By equation (18) the functions $q_k(x)$ and consequently $q_k^{\pm}(\sigma)$ still contain the unknown quantities ε_k, $k=0, 1, \ldots, N-1$, which we now evaluate. By use of the Kelvin circulation theorem we found the relation

$$\varepsilon_l = -j\omega \cdot e^{j\omega} \cdot \int_{-1}^{+1} \gamma_l(x')\, dx' \quad \text{for } l=0, 1, \ldots, N-1.$$

The densities $\gamma_l(x')$ of the profile vortices occurred in the expression for the reduced velocity field $W^*(z)$ in equation (21). Applying the Plemelj formulas [*loc. cit.*] after separating the kernel in (21) into a singular and a regular term, we obtain

$$W^*(z_l[x'] + i \cdot 0) - W^*(z_l[x'] - i \cdot 0) = \gamma_l(x') \tag{56}$$

for $-1 < x' < +1$ and $l=0, 1, \ldots, N-1$. Thus the above formula for ε_l may be rewritten as

$$\varepsilon_l = -j\omega \cdot e^{j\omega} \cdot \int_{-1}^{+1} [W^*(z_l[x'] + i \cdot 0) - W^*(z_l[x'] - i \cdot 0)]\, dx'.$$

The complex velocity field $W^*(z)$ is holomorphic outside the line profiles $z=z_l[x']$ of the staggered cascade and has only integrable singularities on the profiles. By the Cauchy theorem the path of integration may be replaced by a simple contour $\mathscr{L}_l$, which runs around the l^{th} profile only, and no more than once, in the clockwise direction. Hence $\varepsilon_l = -j\omega \cdot e^{j\omega} \cdot \oint_{\mathscr{L}_l} W^*(z)\, dz$, which after conformal mapping becomes $\varepsilon_l = -j\omega \cdot e^{j\omega} \cdot \oint_{\Lambda_l} \Omega(\zeta)\, d\zeta$. Λ_l, the image of $\mathscr{L}_l$, is a contour which merely runs around the segment $\tau \cdot l - 1 \leqq \xi \leqq \tau \cdot l + 1$, $\eta = 0$, and around no other one, once and in the clockwise direction. Relative to the imaginary unit i the integrals $\oint_{\mathscr{L}_l} W^*(z)\, dz$ are real, since only a field with vortices is in question. In evaluating the integrals $\oint_{\Lambda_l} \Omega(\zeta)\, d\zeta$, we may confine ourselves to the field $\Omega_1(\zeta)$, since $\Omega_2(\zeta)$ is a source field with vanishing total source strengths and $\Omega_1^1(\zeta)$ is a homogenous parallel field. After substituting $\zeta = \varrho + \tau \cdot l$, we obtain

$$\varepsilon_l = -j\omega \cdot e^{j\omega} \cdot \oint_{\Lambda_0} \Omega_1(\varrho + \tau \cdot l)\, d\varrho \quad \text{for } l=0, 1, \ldots, N-1. \tag{57}$$

After inserting the term for $\Omega_1(\varrho + \tau \cdot l)$ from (53) into (57), we obtain a system of linear equations in the ε_l, which we shall not write down here because it is too lengthy. By means of the following transformation we shall succeed in separating these equations according to the unknown quantities. This transformation is based on the following conjugate equations: If $f_0(x), \ldots, f_{N-1}(x)$ be any N functions of x, these functions can always be written in the form

$$\begin{aligned} f_k(x) &= \sum_{n=0}^{N-1} f^{(n)}(x) \cdot \exp\left(2\pi i \frac{nk}{N}\right) \quad \text{with} \\ f^{(n)}(x) &= \frac{1}{N} \cdot \sum_{\mu=0}^{N-1} f_\mu(x) \cdot \exp\left(-2\pi i \frac{\mu n}{N}\right) \quad \text{for } k, n=0, 1, \ldots, N-1. \end{aligned} \tag{58}$$

Therefore we shall transform the system of linear equations for the ε_l into a new one by multiplying the l^{th} equation by $\frac{1}{N}\cdot\exp\left(-2\pi i\frac{l\,n}{N}\right)$ and adding the resulting equations. The results of some summations occurring herein are listed in the Appendix. After application of those formulae, the transformed system has the form

$$
\begin{aligned}
-\frac{e^{-j\omega N\tau}\cdot\varepsilon^{(0)}}{j\,\omega} &= \oint_{A_0}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{-\frac12}\cdot\int_{\sigma=-1}^{+1}\sum_{k=0}^{N-1}\Bigg(r_{k1}(\sigma+\tau\cdot k)\times\\
&\times\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac12}\cdot\left\{\frac{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}{\sin\frac{\pi}{\tau}(\varrho-\sigma)}+\frac{\cos\frac{\pi\varrho}{\tau}}{\sin\frac{\pi}{\tau}\cos\lambda+\cos\frac{\pi\zeta_0}{\tau}}\right\}-\\
&-r_{k2}(\sigma+\tau\cdot k)\cdot\frac{\sin\frac{\pi}{\tau}\cdot\cos\lambda\cdot\cos\frac{\pi}{\tau}(\sigma-\zeta_0)}{\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi\zeta_0}{\tau}}\cdot\cos\frac{\pi\varrho}{\tau}\Bigg)\frac{d\sigma\,d\varrho}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)},
\end{aligned}
\tag{59a}
$$

$$
\begin{aligned}
-\frac{e^{-j\omega N\tau}\cdot\varepsilon^{(n)}}{j\,\omega} &= \oint_{A_0}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{-\frac12}\times\\
&\times\int_{\sigma=-1}^{+1}\sum_{k=0}^{N-1}1\cdot\exp\left[-2\pi i\left(\frac{N}{2}-n\right)\frac{\varrho-\sigma}{N}-2\pi i\frac{k\,n}{N}\right]\cdot\Bigg\{r_{k1}(\sigma+\tau\cdot k)\times\\
&\times\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac12}\frac{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}{\sin\frac{\pi}{\tau}(\varrho-\sigma)}-r_{k2}(\sigma+\tau\cdot k)\sin\frac{\pi}{\tau}\cdot\cos\lambda\Bigg\}\frac{d\sigma\,d\varrho}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}
\end{aligned}
\tag{59b}
$$

$$\text{for } n=1,2,\ldots,N-1\,.$$

Since

$$
\begin{aligned}
\frac{1}{N}\cdot\sum_{k=0}^{N-1}q_k^{\pm}(\sigma)\cdot\exp\left(-2\pi i\frac{k\,n}{N}\right) &= \frac{1}{N}\cdot\sum_{k=0}^{N-1}g_k^{\pm}(\sigma)\cdot\exp\left(-2\pi i\frac{k\,n}{N}\right)-\\
-\frac{1}{N}\cdot\sum_{k=0}^{N-1}\sum_{\mu=0}^{N-1}\varepsilon_\mu\cdot h_{k,\mu}^{\pm}(\sigma)\cdot\exp\left(-2\pi i\frac{k\,n}{N}\right) &= g^{\pm(n)}(\sigma)-\\
-\frac{1}{N}\cdot\sum_{\mu=0}^{N-1}\varepsilon_\mu\cdot\exp\left(-2\pi i\frac{n\,\mu}{N}\right)\cdot\sum_{k=0}^{N-1}h_{k,\mu}^{\pm}(\sigma)\cdot\exp\left(-2\pi i\frac{n\,[k-\mu]}{N}\right),
\end{aligned}
$$

taking account of the fact that $h_{k,\mu}^{\pm}(\sigma)=h_r^{\pm}(\sigma)$ for $k-\mu=r+m\cdot N$, m any integer, we obtain the formula

$$q^{\pm(n)}(\sigma)=g^{\pm(n)}(\sigma)-N\cdot\varepsilon^{(n)}\cdot h^{\pm(n)}(\sigma)\tag{60}$$

for $n=0,1,\ldots,N-1$. After changing the orders of the integrations in equations (59b) and substituting $q^{\pm(n)}(\sigma)$ for $r_1^{(n)}(\sigma)$ and $r_2^{(n)}(\sigma)$ according to the

formulas (31) and (34), which were likewise transformed, we may write

$$
\begin{aligned}
-\varepsilon^{(0)}\cdot\frac{e^{-j\omega\cdot 2\tau}}{j\omega\cdot T\cos\lambda} = \int\limits_{\sigma=-1}^{+1}\Bigg\{&\left[g^{+(0)}(\sigma)+g^{-(0)}(\sigma)-N\cdot\varepsilon^{(0)}\cdot\left(h^{+(0)}(\sigma)+h^{-(0)}(\sigma)\right)\right]\times\\
&\times\tan\lambda+\left[g^{+(0)}(\sigma)-g^{-(0)}(\sigma)-N\cdot\varepsilon^{(0)}\cdot\left(h^{+(0)}(\sigma)-h^{-(0)}(\sigma)\right)\right]\frac{\sin\frac{\pi\sigma}{\tau}}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}\Bigg\}\times\\
&\times\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}\cdot\left[J^0_{01}(\sigma)+J^0_{02}(\sigma)\frac{1}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)\left(\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi\zeta_0}{\tau}\right)}\right]\cdot d\sigma-\\
&-\int\limits_{\sigma=-1}^{+1}\Bigg\{\left[g^{+(0)}(\sigma)-g^{-(0)}(\sigma)-N\cdot\varepsilon^{(0)}\left(h^{+(0)}(\sigma)-h^{-(0)}(\sigma)\right)\right]\cdot\tan\lambda+\\
&+\left[g^{+(0)}(\sigma)+g^{-(0)}(\sigma)-N\cdot\varepsilon^{(0)}\left(h^{+(0)}(\sigma)+h^{-(0)}(\sigma)\right)\right]\frac{\sin\frac{\pi\sigma}{\tau}}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}\Bigg\}\times\\
&\times\frac{\sin\frac{\pi}{\tau}\cdot\cos\lambda\cdot\operatorname{ctn}\frac{\pi}{\tau}(\sigma-\zeta_0)}{\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi\zeta_0}{\tau}}\cdot J^0_{02}(\sigma)\,d\sigma
\end{aligned}
\tag{61a}
$$

and

$$
\begin{aligned}
-\varepsilon^{(n)}\cdot\frac{e^{-j\omega\cdot 2\tau}}{j\omega\cdot T\cos\lambda} = \int\limits_{\sigma=-1}^{+1}\left[\sin\frac{\pi}{\tau}(\sigma-\zeta_0)\right]^{-1}\Bigg\{&\left[g^{+(n)}(\sigma)+g^{-(n)}(\sigma)-\right.\\
&\left.-N\cdot\varepsilon^{(n)}\left(h^{+(n)}(\sigma)+h^{-(n)}(\sigma)\right)\right]\cdot\tan\lambda+\left[g^{+(n)}(\sigma)-g^{-(n)}(\sigma)-N\cdot\varepsilon^{(n)}\left(h^{+(n)}(\sigma)-\right.\right.\\
&\left.\left.-h^{-(n)}(\sigma)\right)\right]\frac{\sin\frac{\pi\sigma}{\tau}}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}\Bigg\}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}\cdot J^0_{n1}(\sigma)\cdot d\sigma-\\
&-\int\limits_{\sigma=-1}^{+1}\left[\sin\frac{\pi}{\tau}(\sigma-\zeta_0)\right]^{-1}\Bigg\{\left[g^{+(n)}(\sigma)-g^{-(n)}(\sigma)-N\cdot\varepsilon^{(n)}\left(h^{+(n)}(\sigma)-\right.\right.\\
&\left.\left.-h^{-(n)}(\sigma)\right)\right]\cdot\tan\lambda+\left[g^{+(n)}(\sigma)+g^{-(n)}(\sigma)-N\cdot\varepsilon^{(n)}\cdot\left(h^{+(n)}(\sigma)+h^{-(n)}(\sigma)\right)\right]\times\\
&\times\frac{\sin\frac{\pi\sigma}{\tau}}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}\Bigg\}\sin\frac{\pi}{\tau}\cos\lambda\cdot J^0_{n2}(\sigma)\,d\sigma \qquad \text{for } n=1,2,\ldots,N-1.
\end{aligned}
\tag{61b}
$$

These equations are separated according to the unknowns $\varepsilon^{(n)}$ and can easily be evaluated. The functions $J^0_{n1}(\sigma)$ and $J^0_{n2}(\sigma)$ occurring in the last formulae are given in the Appendix.

In the preceeding formulae we determine the functions $g^{\pm(n)}(\sigma)$ by means of the prescribed velocities of the blade deflections $g_l(x) = g(z_l[x])$ as given by

$$g^{\pm(n)}(\sigma) = \frac{1}{N} \cdot \sum_{l=0}^{N-1} g[f(\sigma + \tau \cdot l \pm i \cdot 0)] \cdot \exp\left(-2\pi i \frac{l n}{N}\right).$$

This is equivalent to the separation of the system modes of the blade row into harmonic fundamental modes over the circumference, as may be found in a paper by F. Lane [22]. For the functions $h^{\pm(n)}(\sigma)$ we obtain the expressions

$$\begin{aligned} h^{\pm(0)}(\sigma) = &-\frac{1}{4TN} \cdot \int_{x'=1}^{\infty} dx' \cdot e^{-j\omega x'} \times \\ &\times \frac{\cos\lambda \cdot \exp\left\{\frac{2\pi}{T}[f(\sigma \pm i\,0) - x'] \cdot \cos\lambda\right\} - \cos\left\{\lambda + \frac{2\pi}{T}[f - x'] \sin\lambda\right\}}{\sinh^2\left\{\frac{\pi}{T}[f(\sigma \pm i\,0) - x'] \cos\lambda\right\} + \sin^2\left\{\frac{\pi}{T}[f - x'] \sin\lambda\right\}}, \end{aligned} \tag{62a}$$

$$\begin{aligned} h^{\pm(n)}(\sigma) = &-\frac{1}{4TN} \cdot \int_{x'=1}^{\infty} e^{-j\omega x'} \cdot \left[\exp\left\{\frac{2\pi i n}{TN}[f(\sigma \pm i\,0) - x'] \cdot \sin\lambda\right\} \times \right. \\ &\times \sinh\left\{\frac{2\pi(N-n)}{TN}[f(\sigma \pm i\,0) - x'] \cdot \cos\lambda\right\} + \\ &+ \exp\left\{-\frac{2\pi i (N-n)}{TN}[f(\sigma \pm i\,0) - x'] \cdot \sin\lambda\right\} \times \\ &\left. \times \sinh\left\{\frac{2\pi n}{TN}[f(\sigma \pm i\,0) - x'] \cdot \cos\lambda\right\}\right] \times \\ &\times \left[\sinh^2\left\{\frac{\pi}{T}[f(\sigma \pm i \cdot 0) - x'] \cdot \cos\lambda\right\} + \right. \\ &\left. + \sin^2\left\{\frac{\pi}{T}[f(\sigma \pm i \cdot 0) - x'] \cdot \sin\lambda\right\}\right]^{-1} \cdot dx' \\ &\text{for } n = 1, 2, \ldots, N-1. \end{aligned} \tag{62b}$$

The constants $\varepsilon^{(n)}$ are evaluated by means of the formulae (61a, b); the values of the constants ε_k, $k = 0, 1, \ldots, N-1$, of the free vortices can be ascertained from equations (59). These values we substitute into formula (53) for the total flow field in the ζ-plane. However, it serves our purpose better if we make use of the Fourier transforms $\varepsilon^{(n)}$, $g^{\pm(n)}(\sigma)$, and $h^{\pm(n)}(\sigma)$ and rewrite formula (53) in terms of these quantities. Because of the relations

$$q_k^{\pm}(\sigma) = \sum_{n=0}^{N-1} q^{\pm(n)}(\sigma) \cdot \exp\left(2\pi i \frac{k n}{N}\right), \qquad k = 0, 1, \ldots, N-1,$$

the summations over k and n in (53) may be interchanged, and the one over k can be carried out by means of the formulae in the Appendix. After multiplying $\Omega(\zeta)$ by $d\zeta/dz$ according to (27), we obtain the required reduced complex velocity field $W^*(z)$ in parametric form as a sum of N single fields corresponding to the N fundamental system modes of the blade row:

$$W^*[z(\zeta)] = \Omega(\zeta) \cdot \frac{d\zeta}{dz} = \sum_{n=0}^{N-1} \Omega^{(n)}(\zeta) \cdot \frac{d\zeta}{dz}$$

with $z = z(\zeta) = f(\zeta)$ given by equation (26) and

$$\Omega^{(0)}(\zeta) = \frac{1}{\tau}\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\zeta}{\tau}\right)^{-\frac{1}{2}} \times$$

$$\times \int\limits_{\sigma=-1}^{+1}\left(r_1^{(0)}(\sigma)\cdot\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}} \left\{\frac{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}{\sin\frac{\pi}{\tau}(\zeta-\sigma)} + \frac{\cos\frac{\pi\zeta}{\tau}}{\sin\frac{\pi}{\tau}\cdot\cos\lambda + \cos\frac{\pi\zeta_0}{\tau}}\right\} - \right.$$

$$\left. (63\text{a}) \quad - r_2^{(0)}(\sigma)\cdot\sin\frac{\pi}{\tau}\cdot\cos\lambda\,\frac{\cos\frac{\pi}{\tau}(\sigma-\zeta_0)\cdot\cos\frac{\pi\zeta}{\tau}}{\sin\frac{\pi}{\tau}\cdot\cos\lambda + \cos\frac{\pi\zeta_0}{\tau}}\right)\frac{d\sigma}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)} +$$

$$+ \frac{1}{\tau}\int\limits_{\sigma=-1}^{+1}\left(r_2^{(0)}(\sigma)\left\{\operatorname{ctn}\frac{\pi}{\tau}(\zeta-\sigma) + \frac{\sin\frac{\pi}{\tau}\cdot\cos\lambda}{\sin\frac{\pi}{\tau}\cdot\cos\lambda + \cos\frac{\pi\zeta_0}{\tau}}\operatorname{ctn}\frac{\pi}{\tau}(\sigma-\zeta_0)\right\} - \right.$$

$$\left. - r_1^{(0)}(\sigma)\cdot\frac{\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}{\left(\sin\frac{\pi}{\tau}\cdot\cos\lambda + \cos\frac{\pi\zeta_0}{\tau}\right)\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}\right) d\sigma,$$

$$\Omega^{(n)}(\zeta) = \frac{1}{\tau}\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\zeta}{\tau}\right)^{-\frac{1}{2}}\cdot\int\limits_{\sigma=-1}^{+1}\left(r_1^{(n)}(\sigma)\cdot\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}\frac{\sin\frac{\pi}{\tau}(\zeta-\zeta_0)}{\sin\frac{\pi}{\tau}(\zeta-\sigma)} - \right.$$

$$\left. (63\text{b}) \quad - r_2^{(n)}(\sigma)\cdot\sin\frac{\pi}{\tau}\cdot\cos\lambda\right)\frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\zeta-\sigma)\right]}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}\, d\sigma +$$

$$+ \frac{1}{\tau}\cdot\int\limits_{\sigma=-1}^{+1} r_2^{(n)}(\sigma)\cdot\frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\zeta-\sigma)\right]}{\sin\frac{\pi}{\tau}(\zeta-\sigma)}\, d\sigma \qquad \text{for } n = 1, 2, \ldots, N-1.$$

In the special case when all blades oscillate in phase, all Fourier transforms of order $n > 0$ vanish.

Thus the reduced problem of perturbation flow has been solved completely, at least in parametric form. In order to obtain the field of the unsteady perturbation flow in the z-plane, we must add the velocity field induced by the free vortices in the wakes $z = z_m[x']$ with $x' \geqq 1$ and m any integer. Finally, we multiply the terms for the amplitudes by $\exp(j\omega s)$ and then take the real part with respect to j. We state the field induced by the free vortices, $W_{\text{fr}}(z)$ as a function of the $\varepsilon^{(n)}$ omitting the time-dependent factor:

$$(64) \qquad \begin{aligned} W_{\text{fr}}(z) &= \frac{i\,e^{i\lambda}}{T N}\sum_{k=0}^{N-1}\varepsilon_k\cdot\int\limits_1^\infty e^{-j\omega x'}\frac{\exp\frac{2\pi e^{i\lambda} z}{TN}}{\exp\frac{2\pi e^{i\lambda} z}{TN} - \exp\frac{2\pi e^{i\lambda}\cdot z_k[x']}{TN}}\, dx', \\ W_{\text{f}1}(z) &= \sum_{n=0}^{N-1}\varepsilon^{(n)}\cdot W_{\text{fr}}^{(N-n)}(z) \end{aligned}$$

with

$$W_{\mathrm{fr}}^{(N)}(z) = \frac{i\,e^{i\lambda}}{T} \int_{x'=1}^{\infty} e^{-j\omega x'} \cdot \frac{\exp \frac{2\pi e^{i\lambda} z}{T}}{\exp \frac{2\pi e^{i\lambda} z}{T} - \exp \frac{2\pi e^{i\lambda} x'}{T}}\, dx', \tag{65a}$$

$$W_{\mathrm{fr}}^{(N-n)}(z) = \frac{i\,e^{i\lambda}}{2T} \int_{x'=1}^{\infty} e^{-j\omega x'} \cdot \frac{\exp\left[\frac{\pi e^{i\lambda}}{TN}(2n-N)(z-x')\right]}{\sinh\left[\frac{\pi e^{i\lambda}}{T}(z-x')\right]}\, dx' \tag{65b}$$

$$\text{for } n = 1, 2, \ldots, N-1.$$

The first term gives the field in the special case when the blades are in synchronized oscillations.

6. Derivation of Formulae for Lift and Moment

Since the perturbation velocity field $U_0 \cdot W(z)$ is known, at least in parametric form, we are able to evaluate the perturbation pressure field $\varrho_0 \cdot U_0^2 \cdot p(x, y, t)$ with the density ϱ_0 of the undisturbed fluid. Due to the fact that the perturbation velocity field must vanish as $x \to -\infty$, we obtain the pressure field

$$p(x, y) = -U(x, y) - j\omega \cdot \int_{-\infty}^{x} U(x', y)\, dx' \tag{66}$$

by means of the linearized Euler equations. For conciseness, we substitute $U_m^{\pm}(x)$ for $U(x + m\sin\lambda,\ m\cos\lambda \pm 0)$ and in the same way $V_m^{\pm}(x)$ and $p_m^{\pm}(x)$. The pressure jump $\Delta p_m(x) = p_m^+(x) - p_m^-(x)$ on the blade profiles $z = z_m[x]$, $-1 < x < +1$, m being any integer, is then given by

$$\Delta p_m(x) = -\gamma_m(x) - j\omega \cdot \int_{-1}^{x} \gamma_m(x')\, dx', \tag{67}$$

$\gamma_m(x) = U_m^+(x) - U_m^-(x)$ being the density of the profile vortices. By integrating over the chord length, we obtain the lift (per unit of length in radial direction of the blade) $\mathscr{A}_m = \frac{1}{2}\varrho_0 \cdot U_0^2 \cdot 2c \cdot A_m$ with

$$A_m = \int_{-1}^{+1} \left[\gamma_m(x') + j\omega \int_{-1}^{x} \gamma_m(x')\, dx'\right] dx. \tag{68}$$

The moment acting on the m^{th} blade relative to the leading edge and considered positive if tail-heavy, is given by $\mathfrak{M}_m = -\frac{1}{2}\varrho_0 \cdot U_0^2 \cdot 2c^2 \cdot M_m$ with

$$M_m = -\int_{-1}^{+1} (x+1) \left[\gamma_m(x) + j\omega \cdot \int_{-1}^{x} \gamma_m(x')\, dx'\right] dx. \tag{69}$$

Since the cascade was obtained by the development of a blade row, the following relations are the consequences: $A_m = A_k$ and $M_m = M_k$ for $m \equiv k(N)$ and $k = 0, \ldots, N-1$. By means of partial integration we may transform the formulae for A_m and M_m into the following:

$$A_k = (1 + j\omega) \cdot \int_{-1}^{+1} \gamma_k(x)\, dx - j\omega \cdot \int_{-1}^{+1} x \cdot \gamma_k(x)\, dx, \tag{70}$$

and

$$M_k = -(1 + \tfrac{3}{2} j\omega) \cdot \int_{-1}^{+1} \gamma_k(x)\, dx - (1 - j\omega) \cdot \int_{-1}^{+1} x \cdot \gamma_k(x)\, dx + \tfrac{1}{2} j\omega \cdot \int_{-1}^{+1} x^2 \cdot \gamma_k(x)\, dx. \tag{71}$$

Due to (12) and the definition of the $z_k[x]$, we may substitute $-\frac{e-j\omega}{j\omega}\cdot\varepsilon_k$ for $\int\limits_1^{+1}\gamma_k(x)\,dx$ and $z_k[x]-iTk\,e^{-i\lambda}$ for x in (70) and (71). Regarding the vortex densities $\gamma_k(x)$ as the differences of the perturbation velocities from the upper to the lower sides of the profiles, *i.e.*, $\gamma_k(x)=W^*\left(z_k[x]+i\cdot 0\right)-W^*\left(z_k[x]-i\cdot 0\right)$, we may transform the integrals in (70) and (71) into contour integrals around the k^{th} profile. By use of the Cauchy integral theorem we are also able to extend the integrations over any simple contour $\mathscr{L}_k$ which runs round the profile just once in the clockwise direction. Then for $k=0, 1, \ldots, N-1$ we deduce

$$(72)\qquad A_k=-\left(\frac{e-j\omega}{j\omega}+e^{-j\omega}\right)\cdot\varepsilon_k-j\omega\cdot\oint\limits_{\mathscr{L}_k}(z-iTk\,e^{-i\lambda})\cdot W^*(z)\,dz,$$

$$(73)\qquad \begin{aligned} M_k=\left(\frac{e-j\omega}{j\omega}+\frac{3}{2}e^{-j\omega}\right)\cdot\varepsilon_k-(1-j\omega)\cdot\oint\limits_{\mathscr{L}_k}(z-iTk\,e^{-i\lambda})\cdot W^*(z)\,dz+\\ +\frac{1}{2}j\omega\cdot\oint\limits_{\mathscr{L}_k}(z-iTk\,e^{-i\lambda})^2\cdot W^*(z)\,dz.\end{aligned}$$

By means of the conformal mapping $z=f(\zeta)$ of the horizontal cascade of line profiles onto the staggered one, we can transform the curves of integration $\mathscr{L}_k$ into simple contours Λ_k which encircle only the k^{th} segment, $\tau\cdot k-1\leqq\xi\leqq\tau\cdot k+1$, $\eta=0$, once in the clockwise direction. If we then put $\zeta=\tau k+\varrho$ and take account of the relation $f(\tau k+\varrho)=iTk e^{-i\lambda}+f(\varrho)$, we obtain for $k=0, 1, \ldots, N-1$

$$(74)\qquad A_k=-\left(\frac{e-j\omega}{j\omega}+e^{-j\omega}\right)\cdot\varepsilon_k-j\omega\cdot\oint\limits_{\Lambda_0}f(\varrho)\cdot\Omega(\varrho+\tau\cdot k)\cdot d\varrho,$$

$$(75)\qquad \begin{aligned} M_k=\left(\frac{e-j\omega}{j\omega}+\frac{3}{2}e^{-j\omega}\right)\cdot\varepsilon_k-(1-j\omega)\cdot\oint\limits_{\Lambda_0}f(\varrho)\cdot\Omega(\varrho+\tau\cdot k)\,d\varrho+\\ +\frac{1}{2}j\omega\oint\limits_{\Lambda_0}[f(\varrho)]^2\,\Omega(\varrho+\tau\cdot k)\,d\varrho,\end{aligned}$$

with $\Omega(\zeta)$ from equation (53).

Immediate calculation of the terms A_k and M_k is not advisable, but it is more suitable to calculate the "Fourier transforms" $A^{(n)}$ and $M^{(n)}$ that correspond to the N fundamental system modes of the blade row. We then apply the transformation given by equation (58) to the formulae (74) and (75) and, after a series of calculations, obtain

$$(76\text{a})\qquad \begin{aligned} A^{(0)}=&-\left(\frac{e-j\omega}{j\omega}+e^{-j\omega}\right)\cdot\varepsilon^{(0)}-\frac{j\omega}{\tau}\cdot\int\limits_{\sigma=-1}^{+1}\left(r_1^{(0)}(\sigma)\cdot\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}J_{01}^1(\sigma)\right)\cdot d\sigma-\\ &-\frac{j\omega}{\tau}\cdot\int\limits_{\sigma=-1}^{+1}\left(r_2^{(0)}(\sigma)\left[J_{03}^1(\sigma)\cdot\sin\frac{\pi}{\tau}(\sigma-\zeta_0)+J_{04}^1\frac{\sin\frac{\pi}{\tau}\cdot\cos\lambda\cdot\cos\frac{\pi}{\tau}(\sigma-\zeta_0)}{\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi\zeta_0}{\tau}}\right]-\right.\\ &\left.-r_1^{(0)}(\sigma)\cdot\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}\frac{J_{04}^1}{\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi}{\tau}\zeta_0}\right)\frac{d\sigma}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)},\end{aligned}$$

$$A^{(n)} = -\left(\frac{e^{-j\omega}}{j\omega} + e^{-j\omega}\right) \cdot \varepsilon^{(n)} - \frac{j\omega}{\tau} \cdot \int_{\sigma=-1}^{+1} \left(r_1^{(n)}(\sigma) \cdot \left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}} J_{n1}^1(\sigma) - \right.$$

$$\text{(76b)} \quad \left. - r_2^{(n)}(\sigma) \cdot \sin\frac{\pi}{\tau} \cdot \cos\lambda \cdot J_{n2}^1(\sigma)\right) \frac{d\sigma}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)} - \frac{j\omega}{\tau} \cdot \int_{\sigma=-1}^{+1} r_2^{(n)}(\sigma) \cdot J_{n3}^1(\sigma)\, d\sigma ,$$

$$\text{for } n = 1, 2, \ldots, N-1 ,$$

$$M^{(0)} = \left(\frac{e^{-j\omega}}{j\omega} + \frac{3}{2} e^{-j\omega}\right) \cdot \varepsilon^{(0)} + \frac{1}{\tau} \cdot \int_{\sigma=-1}^{+1} \left(r_1^{(0)}(\sigma) \cdot \left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}} \left[\left\{ \frac{1}{2} j\omega \cdot J_{01}^2(\sigma) - \right.\right.\right.$$

$$\left. - (1 - j\omega) \cdot J_{01}^1(\sigma) \right\} \cdot \sin\frac{\pi}{\tau}(\sigma - \zeta_0) + \frac{1}{2} j\omega \cdot \frac{J_{02}^2}{\sin\frac{\pi}{\tau} \cdot \cos\lambda + \cos\frac{\pi}{\tau}\zeta_0} \right] -$$

$$\left. - r_2^{(0)}(\sigma) \cdot \frac{1}{2} j\omega \cdot J_{02}^2 \frac{\sin\frac{\pi}{\tau} \cdot \cos\lambda \cdot \cos\frac{\pi}{\tau}(\sigma-\zeta_0)}{\sin\frac{\pi}{\tau} \cdot \cos\lambda + \cos\frac{\pi}{\tau}\zeta_0} \right) \frac{d\sigma}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)} +$$

(77a)

$$+ \frac{1}{\tau} \cdot \int_{\sigma=-1}^{+1} \left(r_2^{(0)}(\sigma) \left[\sin\frac{\pi}{\tau}(\sigma - \zeta_0) \cdot \left\{ \frac{1}{2} j\omega \cdot J_{03}^2(\sigma) - (1 - j\omega) \cdot J_{03}^1(\sigma) \right\} + \right.\right.$$

$$\left. - (1 - j\omega) \cdot J_{04}^1 \cdot \frac{\sin\frac{\pi}{\tau} \cdot \cos\lambda \cdot \cos\frac{\pi}{\tau}(\sigma-\zeta_0)}{\sin\frac{\pi}{\tau} \cos\lambda + \cos\frac{\pi}{\tau}\zeta_0} \right] +$$

$$\left. + r_1^{(0)}(\sigma) \cdot (1 - j\omega) \cdot J_{04}^1 \cdot \frac{\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}{\sin\frac{\pi}{\tau} \cdot \cos\lambda + \cos\frac{\pi}{\tau}\zeta_0} \right) \frac{d\sigma}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)} ,$$

$$M^{(n)} = \left(\frac{e^{-j\omega}}{j\omega} + \frac{3}{2} e^{-j\omega}\right) \cdot \varepsilon^{(n)} +$$

$$+ \frac{1}{\tau} \cdot \int_{\sigma=-1}^{+1} \left(r_1^{(n)}(\sigma) \cdot \left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}} \left\{ \frac{1}{2} j\omega \cdot J_{n1}^2(\sigma) - (1 - j\omega) \cdot J_{n1}^1(\sigma) \right\} - \right.$$

$$\text{(77b)} \quad \left. - r_2^{(n)}(\sigma) \cdot \sin\frac{\pi}{\tau} \cdot \cos\lambda \left\{ \frac{1}{2} j\omega \cdot J_{n2}^2(\sigma) - (1 - j\omega) \cdot J_{n2}^1(\sigma) \right\} \right) \frac{d\sigma}{\sin\frac{\pi}{\tau}(\sigma - \zeta_0)} +$$

$$+ \frac{1}{\tau} \cdot \int_{\sigma=-1}^{+1} r_2^{(n)}(\sigma) \cdot \left\{ \frac{1}{2} j\omega \cdot J_{n3}^2(\sigma) - (1 - j\omega) \cdot J_{n3}^1(\sigma) \right\} \cdot d\sigma$$

$$\text{for } n = 1, 2, \ldots, N-1.$$

In the preceding formulae for $A^{(n)}$ and $M^{(n)}$ the zero term in the denominator for $\sigma=\zeta_0$, which is due to $\sin\frac{\pi}{\tau}(\sigma-\zeta_0)$, is removable by the zero term of the numerator at this point. Thus these are ordinary integrals in a neighborhood of $\sigma=\zeta_0$.

It stands to reason that the above formulae are not very suitable for practical calculations of the forces and moments acting on the line profiles of the staggered cascade. For this purpose appropriate approximations should be derived in order to reduce the extent of the numerical calculations. In this paper, however, we shall not enter into any approximate calculations, since it takes too long to derive them.

7. Appendix

In this section we shall write down without proof the summation and integration formulae needed in the last two sections:

(78a) $$\frac{1}{N}\sum_{l=0}^{N-1} c_0\cdot\exp\left[-\frac{2\pi i n}{N\tau}(\tau\cdot l+\zeta_0)\right]=c_0\cdot\begin{cases}1 & \text{for } n=0\\ 0 & \text{for } n>0,\end{cases}$$

(78b) $$\frac{1}{N}\sum_{l=0}^{N-1}\sum_{\nu=1}^{N/2-1} c_\nu\cdot\exp\left[-\frac{2\pi i n}{N\tau}(\tau\cdot l+\zeta_0)\right]\cdot\cos\left[\frac{2\pi\nu}{N\tau}(\tau\cdot l+\zeta_0)\right]$$
$$=\frac{c_n}{2}\cdot\begin{cases}0 & \text{for } n=0\\ 1 & \text{for } n>0,\end{cases}$$

(78c) $$\frac{1}{N}\sum_{l=0}^{N-1}\sum_{\nu=1}^{N/2-1} d_\nu\cdot\exp\left[-\frac{2\pi i n}{N\tau}(\tau\cdot l+\zeta_0)\right]\cdot\sin\left[\frac{2\pi\nu}{N\tau}(\tau\cdot l+\zeta_0)\right]$$
$$=-\frac{i\cdot d_n}{2}\cdot\begin{cases}0 & \text{for } n=0\\ 1 & \text{for } n>0,\end{cases}$$

(78d) $$\frac{1}{N}\sum_{l=0}^{N-1} c_{N/2}\cdot(-1)^l\cdot\exp\left[-\frac{2\pi i n}{N\tau}(\tau\cdot l+\zeta_0)\right]$$
$$=c_{N/2}\cdot e^{-\frac{\pi i\zeta_0}{\tau}}\cdot\begin{cases}0 & \text{for } n\neq\frac{N}{2}\\ 1 & \text{for } n=\frac{N}{2},\end{cases}$$

(78e) $$\frac{1}{N}\sum_{l=0}^{N-1}\exp\left[-\frac{2\pi i n}{N\tau}(\tau\cdot l+\zeta_0)\right]\cdot\operatorname{ctn}\left[\frac{\pi}{\tau N}(\zeta_0-\sigma-\tau k+\tau l)\right]$$
$$=-\frac{1}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}\cdot\begin{cases}\cos\frac{\pi}{\tau}(\sigma-\zeta_0) & \text{for } n=0\\ \exp\left[\frac{2\pi i}{N\tau}\left\{\left(\frac{N}{2}-n\right)\sigma-\frac{N}{2}\cdot\zeta_0-n\tau k\right\}\right] & \text{for } n>0,\end{cases}$$

(78f) $$\frac{1}{N}\sum_{l=1}^{N-1}\exp\left[-\frac{2\pi i n}{N\tau}(\tau\cdot l+\zeta_0)\right]\cdot(-1)^l\cdot\operatorname{ctn}\left[\frac{\pi}{\tau N}(\zeta_0-\sigma-\tau k+\tau l)\right]$$
$$=-\frac{1}{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}\cdot\begin{cases}(-1)^k\cdot\exp\left[-\frac{2\pi i n}{N\tau}(\sigma+\tau k)\right] & \text{for } n=0,\ldots,\frac{N}{2}-1\\ \cos\frac{\pi}{\tau}(\sigma-\zeta_0)\cdot e^{-\pi i\zeta_0/\tau}, & n=\frac{N}{2},\end{cases}$$

$$\text{(79a)}\quad \begin{gathered}\frac{1}{N}\sum_{l=0}^{N-1}\frac{\sin\frac{\pi}{N\tau}\left[(N-1)\,\varrho+\sigma+k\cdot\tau-l\cdot\tau-N\cdot\zeta_0\right]}{\sin\frac{\pi}{N\tau}\left(\varrho-\sigma+l\cdot\tau-k\cdot\tau\right)}\cdot e^{-2\pi i\frac{nl}{N}}\\ =\frac{1}{\sin\frac{\pi}{\tau}\left(\varrho-\sigma\right)}\cdot\begin{cases}\sin\frac{\pi}{\tau}\left(\sigma-\zeta_0\right) & \text{for } n=0\\ \sin\frac{\pi}{\tau}\left(\varrho-\zeta_0\right)\cdot\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)\left(\varrho-\sigma\right)-\frac{2\pi i k n}{N}\right] & \text{for } n\neq 0,\end{cases}\end{gathered}$$

$$\text{(79b)}\quad \begin{gathered}\frac{1}{N}\sum_{l=0}^{N-1}\frac{\sin\frac{\pi}{N\tau}\left[(N-1)\left(\varrho-\sigma\right)+\tau\cdot k-\tau\cdot l\right]}{\sin\frac{\pi}{N\tau}\left(\varrho-\sigma+l\cdot\tau-k\cdot\tau\right)}\cdot e^{-2\pi i\frac{nl}{N}}\\ =\begin{cases}0 & \text{for } n=0\\ \exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)\left(\varrho-\sigma\right)-\frac{2\pi i k n}{N}\right] & \text{for } n\neq 0,\end{cases}\end{gathered}$$

$$\text{(79c)}\quad \begin{gathered}\frac{1}{N}\sum_{l=0}^{N-1}\cos\frac{\pi\varrho}{\tau}\cdot\left(\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi\zeta_0}{\tau}\right)^{-1}\cdot e^{-2\pi i\frac{nl}{N}}\\ =\begin{cases}\cos\frac{\pi\varrho}{\tau}\cdot\left(\sin\frac{\pi}{\tau}\cdot\cos\lambda+\cos\frac{\pi\zeta_0}{\tau}\right)^{-1} & \text{for } n=0\\ 0 & \text{for } n\neq 0,\end{cases}\end{gathered}$$

$$\text{(79d)}\quad \begin{gathered}\frac{1}{N}\sum_{k=0}^{N-1}\operatorname{ctn}\frac{\pi}{N\tau}\left(\zeta-\sigma-\tau\cdot k\right)\cdot e^{2\pi i\frac{kn}{N}}\\ =\frac{1}{\sin\frac{\pi}{\tau}\left(\zeta-\sigma\right)}\begin{cases}\cos\frac{\pi}{\tau}\left(\zeta-\sigma\right) & \text{for } n=0\\ \exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)\left(\zeta-\sigma\right)\right] & \text{for } n\neq 0.\end{cases}\end{gathered}$$

In the formulae (76a, b) and (77a, b) the following functions given in integral form occur: $J^{\mu}_{n1}(\sigma)$, $J^{\mu}_{n2}(\sigma)$, $J^{\mu}_{n3}(\sigma)$; $n=0, 1, \ldots, N-1$, and J^{μ}_{04}; $\mu=1, 2$. As our path of integration Λ_0 we may choose the following one due to CAUCHY'S integral theorem and due to the fact that all the integrands of the above mentioned integrals have integrable singularities in $\varrho=\pm 1$. Starting on the upper side of the slit $-1\leqq \operatorname{Re}\varrho\leqq +1$, $\operatorname{Im}\varrho=0$, we go from $\varrho=-1$ to the point $\varrho=\sigma-\varepsilon$ ($\varepsilon>0$), then along a semicircle of the radius ε around $\varrho=\sigma$ to $\varrho=\sigma+\varepsilon$ in the clockwise direction, and then along the upper side of the slit as far as $\varrho=+1$. Then we go back on the lower side of the segment as far as $\varrho=+1$. Now we take our path along the semicircle of the radius ε around $\varrho=\sigma$ in the clockwise direction as far as $\varrho=\sigma-\varepsilon$ and return to the starting point $\varrho=-1$ on the lower side. Finally we let ε tend to zero. If the integrand has a non-integrable singularity at $\varrho=\sigma$, the integral must be understood in the sense of a Cauchy principal value. After a series of calculations which we omit here, we obtain the formulae ($|\sigma|<1$)

$$\text{(80a)}\qquad J^{0}_{01}(\sigma)=\oint_{\Lambda_0}\frac{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{-\frac{1}{2}}}{\sin\frac{\pi}{\tau}\left(\varrho-\sigma\right)}\,d\varrho=0 \qquad \left(\text{NICKEL [23], p. 58}\right),$$

$$\text{(80b)}\qquad J^{0}_{02}(\sigma)=\oint_{\Lambda_0}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{-\frac{1}{2}}\cdot\cos\frac{\pi\varrho}{\tau}\cdot d\varrho=2,$$

$$J_{n1}^{0}(\sigma) = \oint_{\Lambda_0} \frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]\cdot\sin\frac{\pi}{\tau}(\varrho-\zeta_0)}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}\cdot\sin\frac{\pi}{\tau}(\varrho-\sigma)}\,d\varrho$$

(80c)

$$= 2\cdot\int_{-1}^{+1} \frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]\cdot\sin\frac{\pi}{\tau}(\varrho-\zeta_0)}{{}_{+}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}\cdot\sin\frac{\pi}{\tau}(\varrho-\sigma)}\,d\varrho,$$

$$J_{n2}^{0}(\sigma) = \oint_{\Lambda_0} \frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]\cdot d\varrho}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}$$

(80d)

$$= 2\cdot\exp\left[\frac{2\pi i}{N\tau}\sigma\left(\frac{N}{2}-n\right)\right]\cdot P_{-n/N}\left(\cos\frac{2\pi}{\tau}\right) \qquad ([24], \text{p. } 87).$$

Here $P_\nu(t)$ is the Legendre function of the first kind. Due to the relation

(81) $$f(\varrho - i\cdot 0) = -f(\varrho + i\cdot 0) + 2T\cdot\sin\lambda\cdot\frac{\varrho}{\tau},$$

we obtain

$$J_{01}^{1}(\sigma) = \oint_{\Lambda_0} f(\varrho)\cdot\frac{d\varrho}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}\cdot\sin\frac{\pi}{\tau}(\varrho-\sigma)}$$

(82a)

$$= 2T\sin\frac{\lambda}{\tau}\cdot\int_{-1}^{+1}\frac{\varrho\cdot d\varrho}{{}_{+}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}\cdot\sin\frac{\pi}{\tau}(\varrho-\sigma)} +$$

$$+\frac{2iT\cos\lambda}{\pi}\cdot\frac{\log_H\left[\dfrac{\cos\frac{\pi\sigma}{\tau}+{}_{+}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}{\cos\frac{\pi}{\tau}}\right]}{{}_{+}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}},$$

$$J_{01}^{2}(\sigma) = \oint_{\Lambda_0}[f(\varrho)]^2\cdot\frac{d\varrho}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}\cdot\sin\frac{\pi}{\tau}(\varrho-\sigma)}$$

$$= \frac{2T\cdot\cos^2\lambda}{\pi^2}\cdot\int_{-1}^{+1}\left\{\log_H^2\left[\frac{\cos\frac{\pi\varrho}{\tau}+{}_{+}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}{\cos\frac{\pi}{\tau}}\right]+\left(\frac{\pi}{\tau}\right)^2\cdot\operatorname{tg}^2\lambda\cdot\varrho^2\right\}\times$$

(82b)

$$\times\frac{d\varrho}{\sin\frac{\pi}{\tau}(\varrho-\sigma)\cdot{}_{+}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}} +$$

$$+\frac{2iT^2\cdot\sin 2\lambda}{\tau\cdot\pi}\cdot\frac{\sigma\cdot\log_H\left[\dfrac{\cos\frac{\pi\sigma}{\tau}+{}_{+}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}{\cos\frac{\pi}{\tau}}\right]}{{}_{+}\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}},$$

$$\text{(82c)}\quad J_{02}^1 = \oint_{\Lambda_0} f(\varrho) \cdot \frac{\cos\frac{\pi\varrho}{\tau} \cdot d\varrho}{\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}} = 0,$$

$$J_{02}^2 = \oint_{\Lambda_0} [f(\varrho)]^2 \cdot \frac{\cos\frac{\pi\varrho}{\tau} \cdot d\varrho}{\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}$$

$$\text{(82d)}\quad = \frac{2T\cdot\cos^2\lambda}{\pi^2} \cdot \int_{-1}^{+1} \left\{ \log_H^2 \left[\frac{\cos\frac{\pi\varrho}{\tau} + {}_{+}\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}{\cos\frac{\pi}{\tau}} \right] + \left(\frac{\pi}{\tau}\right)^2 \cdot \mathrm{tg}^2\lambda\cdot\varrho^2 \right\} \times$$

$$\times \frac{\cos\frac{\pi\varrho}{\tau} \cdot d\varrho}{\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}},$$

$$J_{03}^1(\sigma) = \oint_{\Lambda_0} f(\varrho) \cdot \mathrm{ctn}\,\frac{\pi}{\tau}(\varrho - \sigma)\, d\varrho$$

$$\text{(82e)}\quad = -\frac{2T\cdot\cos\lambda}{\pi} \int_{-1}^{+1} \log_H \left[\frac{\cos\frac{\pi\varrho}{\tau} + {}_{+}\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}{\cos\frac{\pi}{\tau}} \right] \times$$

$$\times\, \mathrm{ctn}\,\frac{\pi}{\tau}(\varrho - \sigma)\, d\varrho - 2iT\sin\lambda\cdot\frac{\sigma}{\tau},$$

$$J_{03}^2(\sigma) = \oint_{\Lambda_0} [f(\varrho)]^2 \cdot \mathrm{ctn}\,\frac{\pi}{\tau}(\varrho - \sigma)\, d\varrho$$

$$= -\frac{2T^2}{\tau\pi}\sin 2\lambda \cdot \int_{-1}^{+1} \varrho\cdot\log_H \left[\frac{\cos\frac{\pi\varrho}{\tau} + {}_{+}\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}{\cos\frac{\pi}{\tau}} \right] \times$$

$$\text{(82f)}\quad \times\, \mathrm{ctn}\,\frac{\pi}{\tau}(\varrho - \sigma)\, d\varrho\, -$$

$$-\frac{2iT^2\cdot\cos^2\lambda}{\pi^2} \cdot \left\{ \log_H \left[\frac{\cos\frac{\pi\sigma}{\tau} + {}_{+}\left(\sin^2\frac{\pi}{\tau} - \sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}{\cos\frac{\pi}{\tau}} \right] + \left(\frac{\pi}{\tau}\right)^2 \cdot \mathrm{tg}^2\lambda\cdot\sigma^2 \right\},$$

$$\text{(82g)}\quad J_{04}^1 = \oint_{\Lambda_0} f(\varrho)\, d\varrho = \frac{2T}{\pi}\cos\lambda\cdot\log\left(\cos\frac{\pi}{\tau}\right) \qquad \text{([23], p. 113)},$$

$$\text{(82h)}\quad J_{04}^2 = \oint_{\Lambda_0} [f(\varrho)]^2\, d\varrho = 0,$$

$$J^1_{n1}(\sigma) = \oint\limits_{\Lambda_0} f(\varrho) \cdot \frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]\cdot \sin\frac{\pi}{\tau}(\varrho-\zeta_0)}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}\cdot\sin\frac{\pi}{\tau}(\varrho-\sigma)}\,d\varrho$$

(82i)

$$= 2T\,\frac{\sin\lambda}{\tau}\cdot\int\limits_{-1}^{+1}\frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]\cdot\varrho\cdot\sin\frac{\pi}{\tau}(\varrho-\zeta_0)}{{}_+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}\cdot\sin\frac{\pi}{\tau}(\varrho-\sigma)}\,d\varrho +$$

$$+\frac{2iT}{\pi}\cos\lambda\cdot\frac{\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}{{}_+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}\cdot\log_H\left[\frac{\cos\frac{\pi\sigma}{\tau}+{}_+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}{\cos\frac{\pi}{\tau}}\right]$$

for $n = 1, \ldots, N$,

$$J^2_{n1}(\sigma) = \oint\limits_{\Lambda_0}[f(\varrho)]^2\cdot\frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]\cdot\sin\frac{\pi}{\tau}(\varrho-\zeta_0)}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}\cdot\sin\frac{\pi}{\tau}(\varrho-\sigma)}\,d\varrho$$

(82j)

$$=\frac{2T^2}{\pi^2}\cdot\cos^2\lambda\cdot\int\limits_{-1}^{+1}\left\{\log^2_H\left[\frac{\cos\frac{\pi\varrho}{\tau}+{}_+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}{\cos\frac{\pi}{\tau}}\right]+\left(\frac{\pi}{\tau}\right)^2\cdot\operatorname{tg}^2\lambda\cdot\varrho^2\right\}\times$$

$$\times\frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]\cdot\sin\frac{\pi}{\tau}(\varrho-\zeta_0)}{{}_+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}\cdot\sin\frac{\pi}{\tau}(\varrho-\sigma)}\,d\varrho+\frac{2i\,T^2}{\pi\tau}\cdot\sin 2\lambda\times$$

$$\times\frac{\sigma\cdot\sin\frac{\pi}{\tau}(\sigma-\zeta_0)}{{}_+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)}\cdot\log_H\left[\frac{\cos\frac{\pi\sigma}{\tau}+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}}{\cos\frac{\pi}{\tau}}\right],$$

$$J^1_{n2}(\sigma)=\oint\limits_{\Lambda_0}f(\varrho)\cdot\frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]}{{}_+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}\,d\varrho$$

(82k) $$=2T\,\frac{\sin\lambda}{\tau}\cdot\int\limits_{-1}^{+1}\frac{\varrho\cdot\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]}{{}_+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}\,d\varrho$$

for $n = 1, \ldots, N-1$,

$$J^2_{n2}(\sigma)=\oint\limits_{\Lambda_0}[f(\varrho)]^2\cdot\frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]}{\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}\,d\varrho$$

(82l)

$$=\frac{4T^2}{\pi^2}\cdot\cos^2\lambda\cdot\exp\left[\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)\sigma\right]\cdot\int\limits_0^1\frac{\cos\frac{2\pi}{N\tau}\left(\frac{N}{2}-n\right)\varrho}{{}_+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}\times$$

(82l)
$$\times\left\{\log_H^2\left[\frac{\cos\frac{\pi\varrho}{\tau}+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}_{+}}{\cos\frac{\pi}{\tau}}\right]+\left(\frac{\pi}{\tau}\right)^2\cdot \mathrm{tg}^2\lambda\cdot\varrho^2\right\}d\varrho$$

$$\text{for}\quad n=1,\ldots,N-1,$$

$$J^1_{n3}(\sigma)=\oint_{A_0} f(\varrho)\cdot\frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]}{\sin\frac{\pi}{\tau}(\varrho-\sigma)}d\varrho$$

(82m)
$$=-\frac{2T}{\pi}\cdot\cos\lambda\cdot\int_{-1}^{+1}\log_H\left[\frac{\cos\frac{\pi\varrho}{\tau}+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}}{\cos\frac{\pi}{\tau}}\right]\times$$
$$\times\frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]}{\sin\frac{\pi}{\tau}(\varrho-\sigma)}d\varrho-2i\,T\cdot\sin\lambda\cdot\frac{\sigma}{\tau}$$

$$\text{for}\quad n=1,\ldots,N-1,$$

$$J^2_{n3}(\sigma)=\oint_{A_0}[f(\varrho)]^2\cdot\frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]}{\sin\frac{\pi}{\tau}(\varrho-\sigma)}d\varrho$$

(82n)
$$=-\frac{2T^2}{\tau\pi}\cdot\sin 2\lambda\cdot\int_{-1}^{+1}\varrho\cdot\log_H\left[\frac{\cos\frac{\pi\varrho}{\tau}+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\varrho}{\tau}\right)^{\frac{1}{2}}_{+}}{\cos\frac{\pi}{\tau}}\right]\times$$
$$\times\frac{\exp\left[-\frac{2\pi i}{N\tau}\left(\frac{N}{2}-n\right)(\varrho-\sigma)\right]}{\sin\frac{\pi}{\tau}(\varrho-\sigma)}d\varrho-\frac{2i\,T^2}{\pi^2}\cdot\cos^2\lambda\times$$
$$\times\left\{\log_H^2\left[\frac{\cos\frac{\pi\sigma}{\tau}+\left(\sin^2\frac{\pi}{\tau}-\sin^2\frac{\pi\sigma}{\tau}\right)^{\frac{1}{2}}_{+}}{\cos\frac{\pi}{\tau}}\right]+\left(\frac{\pi}{\tau}\right)^2\cdot\mathrm{tg}^2\lambda\cdot\sigma^2\right\}$$

$$\text{for}\quad n=1,\ldots,N-1,$$

with the real logarithm $\log_H \ldots$.

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Zur mathematischen Theorie akustischer Wellenfelder

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Einleitung

Die vorliegende Untersuchung eröffnet eine Reihe von drei Arbeiten, in denen Fragen der mathematischen Theorie stationärer akustischer Wellenfelder in inhomogenen Medien behandelt werden. Die akustischen Eigenschaften eines flüssigen oder gasförmigen Mediums lassen sich durch die Dichteverteilung $\varrho_0(\mathfrak{x})$ im Ruhezustand, die Schallgeschwindigkeitsverteilung $c(\mathfrak{x})$ und eine nicht negative Dämpfungskonstante a, deren Bedeutung in §1 erläutert wird, charakterisieren.

Die mathematische Beschreibung akustischer Schwingungsvorgänge führt auf die Gleichung

$$\varrho_0 \nabla\left(\frac{1}{\varrho_0} \nabla \Phi\right) = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} + \frac{a}{c^2} \frac{\partial \Phi}{\partial t} - \frac{1}{c^2} \frac{\partial \Psi}{\partial t}, \tag{1}$$

die sich aus den Grundgleichungen der Hydrodynamik durch einen Linearisierungsprozeß ergibt. Hierbei ist $\Psi(\mathfrak{x}, t)$ das Potential einer vorgegebenen konservativen Kraftdichteverteilung $\mathfrak{K}(\mathfrak{x}, t)$. Die für das Schallfeld charakteristischen Größen sind die Geschwindigkeitsverteilung $\mathfrak{v}(\mathfrak{x}, t)$ und die Druckverteilung $p(\mathfrak{x}, t)$. Sie berechnen sich aus Φ durch

$$\mathfrak{v} = \frac{1}{\varrho_0} \nabla \Phi \tag{2}$$

und

$$p - p_0 = -\frac{\partial \Phi}{\partial t} - a\Phi + \Psi, \tag{3}$$

wobei p_0 der Druck im Ruhezustand ist.

Im folgenden setzen wir voraus, daß $\mathfrak{K}(\mathfrak{x}, t)$ zeitharmonisch mit der Frequenz ω ist, und beschränken uns auf die Diskussion derjenigen Lösungen Φ von (1), die die Gestalt

$$\Phi(\mathfrak{x}, t) = U_1(\mathfrak{x}) \cos \omega t + U_2(\mathfrak{x}) \sin \omega t \tag{4}$$

besitzen. Sie charakterisieren die eingeschwungenen „stationären" akustischen Schwingungsvorgänge. Setzen wir $U = U_1 + i U_2$, so läßt sich (4) auch in der Form

$$\Phi(\mathfrak{x}, t) = \operatorname{Re}\{U(\mathfrak{x})\, e^{-i\omega t}\} \tag{5}$$

schreiben. Die Lösungen Φ von (1) der Gestalt (5) entsprechen eindeutig umkehrbar den Lösungen der zeitunabhängigen Gleichung

$$\varrho_0 \nabla\left(\frac{1}{\varrho_0} \nabla U\right) + \varkappa^2 U = f \tag{6}$$

mit

(7) $$\varkappa^2 = \frac{\omega}{c^2}(\omega + i\,a)$$

und

(8) $$\Psi(\mathfrak{x}, t) = -\frac{c^2}{\omega}\,\mathrm{Re}\{i\,f(\mathfrak{x})\,e^{-i\omega t}\}.$$

Hierbei kann $\varkappa(\mathfrak{x})$ so gewählt werden, daß

(9) $$0 \leqq \operatorname{arc}\varkappa(\mathfrak{x}) < \tfrac{1}{4}\pi$$

gilt.

Im Fall eines homogenen Mediums, auf das keine äußeren Kräfte wirken, verschwindet f, während ϱ_0 und c konstant sind, so daß (6) in die Helmholtzsche Schwingungsgleichung

(10) $$\Delta U + \varkappa^2 U = 0$$

mit konstantem $\varkappa$ übergeht. SOMMERFELD erkannte, daß genau diejenigen Lösungen von (10) physikalisch realisierbaren Schwingungsvorgängen entsprechen, die für $r \to \infty$ und $|\mathfrak{x}_0| = 1$ die asymptotischen Relationen

(11) $$\frac{\partial}{\partial r} U(r\,\mathfrak{x}_0) - i\,\varkappa\,U(r\,\mathfrak{x}_0) = o\left(\frac{1}{r}\right),$$

(12) $$U(r\,\mathfrak{x}_0) = O\left(\frac{1}{r}\right)$$

gleichmäßig für alle Richtungen $\mathfrak{x}_0$ erfüllen (Sommerfeldsche Ausstrahlungsbedingung) [*15*]. Wir setzen voraus, daß alle betrachteten Medien im Äußeren einer genügend großen Kugel $|\mathfrak{x}| = R$ homogen sind und daß f für $|\mathfrak{x}| > R$ verschwindet. Dann können die Bedingungen (11), (12) zur Charakterisierung der physikalisch realisierbaren Lösungen von (6) übernommen werden.

Die Berechnung des von einer zeitharmonischen Kraftdichteverteilung in einem unbegrenzten Medium erzeugten stationären Wellenfelds führt auf folgende Aufgabe:

(A) Es ist eine Funktion U zu bestimmen, die im gesamten Raum der Gleichung (6) und für $r \to \infty$ den Bedingungen (11), (12) genügt.

Im Fall, daß ϱ_0 und $\varkappa$ konstant sind und f hölderstetig ist, besitzt Aufgabe (A) genau eine Lösung, die durch

(13) $$U(\mathfrak{x}) = -\frac{1}{4\pi}\int\limits_{|\mathfrak{x}'|<R} f(\mathfrak{x}')\,\frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|}\,dV_{\mathfrak{x}'}$$

dargestellt wird.

Außer durch zeitharmonische äußere Kräfte kann ein stationäres akustisches Wellenfeld in einem flüssigen oder gasförmigen Medium M durch einen in M eingebetteten festen Körper K erzeugt werden, der elastische Schwingungen mit der Frequenz ω vollführt. Auf der Grenzfläche F zwischen K und M stimmen die Normalgeschwindigkeiten der Randpunkte von K und M überein. Die Berechnung des von K und einer zusätzlich gegebenen zeitharmonischen Kraftdichteverteilung f erzeugten stationären Wellenfelds verlangt die Lösung folgender Aufgabe:

(B) Es ist eine Funktion U zu bestimmen, die im Äußeren von F der Gleichung (6) und für $r \to \infty$ den Bedingungen (11), (12) genügt und deren Normalableitung auf F vorgegebene Werte annimmt (Neumannsches Außenraumproblem).

Mathematisch gleichwertig, jedoch physikalisch von geringerer Bedeutung ist das Dirichletsche Außenraumproblem, bei dem auf F die Werte von U vorgegeben sind. Ihm entspricht physikalisch die Vorgabe der Druckverteilung auf F. Die formulierten Außenraumprobleme sind für die Schwingungsgleichung (10) von Cl. Müller [*11*] und R. Leis [*7*] mit Hilfe der Kapazitätsmethode gelöst worden. Die in den zitierten Arbeiten dargestellten Ergebnisse ermöglichen es im Fall eines homogenen Mediums, die durch pulsierende feste Körper erzeugten stationären akustischen Schwingungsvorgänge mathematisch zu beherrschen.

Die vorliegende Arbeit und die beiden geplanten weiteren Arbeiten setzen es sich zum Ziel, die Methoden zur Untersuchung stationärer akustischer Wellenfelder auf inhomogene Medien auszudehnen. Hierzu sollen zunächst die Aufgaben (A) und (B) für hölderstetige Funktionen $\varkappa$ und f und hölderstetig differenzierbare Funktionen ϱ_0 unter der Voraussetzung gelöst werden, daß $\varkappa$ und ϱ_0 für $|\mathfrak{x}| > R$ konstant sind und f für $|\mathfrak{x}| > R$ verschwindet.

Darüber hinaus wollen wir Methoden entwickeln, die es ermöglichen, die an den Grenzflächen verschiedener Medien auftretenden Reflexions- und Brechungserscheinungen zu behandeln. Als charakteristisches Beispiel betrachten wir eine zweimal stetig differenzierbare, reguläre, geschlossene Fläche F', die den Raum in das Außengebiet G_a und das Innengebiet G_i zerlegt, und nehmen an, daß G_a durch ein Medium M_a mit der Schallgeschwindigkeit $c_a(\mathfrak{x})$, der Dichte $\varrho_{0a}(\mathfrak{x})$ und der Dämpfungskonstanten a_a und G_i durch ein Medium M_i mit den entsprechenden Größen $c_i(\mathfrak{x})$, $\varrho_{0i}(\mathfrak{x})$ und a_i ausgefüllt wird. Wir setzen voraus, daß die Funktionen ϱ_{0a}, ϱ_{0i} und c_a, c_i in ihren Definitionsbereichen $G_a + F'$ und $G_i + F'$ hölderstetig differenzierbar bzw. hölderstetig sind und daß ϱ_{0a} und c_a für $|\mathfrak{x}| > R$ konstant sind. Die Berechnung des durch die Kraftdichteverteilung f in den Medien M_a und M_i erzeugten stationären Wellenfelds führt zu folgender Aufgabe:

(C) Es ist eine Funktion U mit folgenden Eigenschaften zu bestimmen:

a) U erfüllt in G_a und in G_i die Gleichung (6) mit

$$\varrho_0(\mathfrak{x}) = \begin{cases} \varrho_{0a}(\mathfrak{x}) & \text{für} \quad \mathfrak{x} \in G_a \\ \varrho_{0i}(\mathfrak{x}) & \text{für} \quad \mathfrak{x} \in G_i, \end{cases} \qquad \varkappa(\mathfrak{x}) = \begin{cases} \dfrac{\omega(\omega + i\, a_a)}{c_a(\mathfrak{x})^2} & \text{für} \quad \mathfrak{x} \in G_a \\[2ex] \dfrac{\omega(\omega + i\, a_i)}{c_i(\mathfrak{x})^2} & \text{für} \quad \mathfrak{x} \in G_i. \end{cases}$$

b) U genügt für $r \to \infty$ den Bedingungen (11), (12).

c) U ist in $G_a + F'$ und in $G_i + F'$ stetig differenzierbar, und es gilt für $\mathfrak{x} \in F'$

$$(a_a - i\,\omega)\, U_a + i\, \frac{c_a^2}{\omega}\, f = (a_i - i\,\omega)\, U_i + i\, \frac{c_i^2}{\omega}\, f$$

$$\frac{1}{\varrho_{0a}} \frac{\partial U_a}{\partial n} = \frac{1}{\varrho_{0i}} \frac{\partial U_i}{\partial n}.$$

Die Bedingung c), die in § 2 ausführlich erläutert wird, ergibt sich aus der physikalischen Forderung, daß die Druckverteilung und die Normalkomponente der Geschwindigkeit bei dem Durchgang durch die Grenzfläche F' stetig sind. Entsprechend kann Aufgabe (B) verallgemeinert werden.

Die vorliegende Arbeit beginnt mit einer ausführlichen Darstellung der physikalischen Grundlagen, die neben der Herleitung der Gleichung (6) und der zusätzlichen Bedingungen für U an Rand- und Grenzflächen eine eingehende Diskussion der Energieverhältnisse enthält. In § 3 werden einige Lösungen der Schwingungsgleichung (10) diskutiert. Insbesondere werden diejenigen Lösungen von (10) charakterisiert, die den Ausstrahlungsbedingungen (11), (12) genügen. In § 4 wird gezeigt, daß die Aufgaben (A), (B) und (C) höchstens eine Lösung besitzen. Die Argumentation, die in den Eindeutigkeitsbeweisen verwendet wird, steht in engem Zusammenhang mit den in § 1 und § 2 durchgeführten energetischen Betrachtungen. Die Arbeit schließt mit dem Existenznachweis für die Lösung der Aufgabe (A).

In der geplanten zweiten Arbeit werden nach dem Hilbertschen Ansatz der Parametrix die Neumannsche und die Dirichletsche Außenraumaufgabe für die Gleichung (6) gelöst, während im Mittelpunkt der dritten Arbeit der Existenzbeweis für die Aufgabe (C) steht. Die Existenzuntersuchungen können mit Hilfe potentialtheoretischer Methoden auf die Diskussion von Integralgleichungen zurückgeführt werden. Die auftretenden Integralgleichungen und Integralgleichungssysteme lassen sich unter dem einheitlichen Gesichtspunkt der Theorie der linearen Transformationen in Banach-Räumen behandeln.

Es wird eine möglichst in sich geschlossene Darstellung angestrebt, die vornehmlich in der ersten Arbeit die Beziehungen zwischen den physikalischen Grundlagen und ihrer mathematischen Formulierung betont. Es erscheint daher zweckmäßig, auch einige in der Literatur bereits bekannte Dinge aufzunehmen, die jedoch bisher noch keine zusammenhängende und unseren Zwecken völlig entsprechende Darstellung gefunden haben.

Für die Anregung zu dieser Arbeit und für zahlreiche wertvolle Ratschläge möchte ich meinem verehrten Lehrer, Herrn Professor Dr. CLAUS MÜLLER, herzlich danken.

§ 1. Physikalische Grundlagen

Wir betrachten zunächst ein unbegrenztes flüssiges oder gasförmiges Kontinuum, in dem eine Strömung mit der Geschwindigkeitsverteilung $\mathfrak{v}(\mathfrak{x}, t)$, der Druckverteilung $p(\mathfrak{x}, t)$ und der Dichteverteilung $\varrho(\mathfrak{x}, t)$ herrscht; $\mathfrak{v}$, p und ϱ seien stetig differenzierbar.

Wir setzen voraus, daß die Strömung im Endlichen quellfrei ist. Für jedes endliche reguläre* Gebiet G mit der Randfläche F gilt also

$$(1.1)\qquad -\frac{d}{dt}\int_G \varrho(\mathfrak{x}, t)\, dV = \int_F \varrho(\mathfrak{x}, t)\, \mathfrak{v}(\mathfrak{x}, t)\, \mathfrak{n}(\mathfrak{x})\, dF,$$

wobei $\mathfrak{n}(\mathfrak{x})$ der ins Äußere von F weisende Normalenvektor von F im Flächenpunkt $\mathfrak{x}$ ist. Nach dem Satz von GAUSS ergibt sich hieraus die Kontinuitätsgleichung

$$(1.2)\qquad \frac{\partial \varrho}{\partial t} + \nabla(\varrho\, \mathfrak{v}) = 0.$$

Um weitere Beziehungen zwischen $\mathfrak{v}$, p und ϱ zu gewinnen, betrachten wir den Flüssigkeitspunkt P, der sich zur Zeit t in $\mathfrak{x}$ befindet. Die Bahn $\mathfrak{z}(\tau)$ von P

* „regulär" im Sinn von [6], S. 112, 113

erhalten wir, indem wir die zu den Anfangsbedingungen

(1.3) $$\mathfrak{z}(t) = \mathfrak{x}$$

gehörende Lösung des Differentialgleichungssystems

(1.4) $$\mathfrak{z}'(\tau) = \mathfrak{v}\big(\mathfrak{z}(\tau), \tau\big)$$

bestimmen. Es sei $A(\mathfrak{z}, \tau)$ eine stetig differenzierbare Funktion. Der zeitliche Verlauf von A längs der Bahn $\mathfrak{z}(\tau)$ des Flüssigkeitspunktes P wird durch die Funktion

(1.5) $$F(\tau) = A\big(\mathfrak{z}(\tau), \tau\big)$$

beschrieben. Wir bezeichnen daher $F'(t)$ als die substantielle Ableitung der Funktion A im Punkt $\mathfrak{x}$ zur Zeit t. Nach (1.3) und (1.4) ist

(1.6) $$F'(t) = \frac{\partial}{\partial t} A(\mathfrak{x}, t) + \mathfrak{v}(\mathfrak{x}, t)\, \nabla_{\mathfrak{x}} A(\mathfrak{x}, t)\,.$$

Führen wir für die substantielle Ableitung das Symbol D/Dt ein, so gilt nach (1.6)

(1.7) $$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathfrak{v}\, \nabla_{\mathfrak{x}})\,.$$

Für die Beschleunigung des zur Zeit t in $\mathfrak{x}$ befindlichen Flüssigkeitspunktes erhalten wir z.B.

(1.8) $$\mathfrak{b}(\mathfrak{x}, t) = \frac{D}{Dt} \mathfrak{v}(\mathfrak{x}, t) = \frac{\partial \mathfrak{v}}{\partial t} + (\mathfrak{v}\, \nabla)\, \mathfrak{v}\,.$$

Zur Aufstellung der Bewegungsgleichungen betrachten wir ein Flüssigkeitsteilchen μ, das zur Zeit t das reguläre Gebiet G mit der Randfläche F ausfüllt. Die in der Flüssigkeit auftretenden inneren Kräfte setzen sich aus Druck- und Reibungskräften zusammen. Die Druckkräfte lassen sich mit Hilfe der Druckverteilung $p(\mathfrak{x}, t)$ beschreiben, und zwar wirkt auf μ zur Zeit t die Druckkraft

(1.9) $$-\int_F p(\mathfrak{x}, t)\, \mathfrak{n}(\mathfrak{x})\, dF = -\int_G \nabla_{\mathfrak{x}} p(\mathfrak{x}, t)\, dV.$$

Bei akustischen Vorgängen treten nur kleine Geschwindigkeiten und Dichteschwankungen auf. Wir können daher von der Annahme ausgehen, daß auf jedes Flüssigkeitselement eine entgegengesetzt zu seiner Bahn gerichtete Reibungskraft wirkt, die proportional zu seiner Masse und seiner Geschwindigkeit ist. Auf μ wirkt demnach die Reibungskraft

(1.10) $$-a \int_G \varrho(\mathfrak{x}, t)\, \mathfrak{v}(\mathfrak{x}, t)\, dV,$$

wobei a eine nur von dem betreffenden Medium abhängige positive Zahl ist.

Bei der Beschreibung der äußeren Kräfte nehmen wir an, daß es eine stetige Kraftdichteverteilung $\mathfrak{K}(\mathfrak{x}, t)$ gibt, mit deren Hilfe sich die zur Zeit t auf μ wirkende äußere Kraft in der Form

(1.11) $$\int_G \mathfrak{K}(\mathfrak{x}, t)\, dV$$

darstellen läßt. Aus den Newtonschen Grundgleichungen folgt

(1.12) $$\int_G \varrho(\mathfrak{x}, t) \frac{D}{Dt} \mathfrak{v}(\mathfrak{x}, t)\, dV = \int_G \left[\mathfrak{K}(\mathfrak{x}, t) - \nabla p(\mathfrak{x}, t) - a\, \varrho(\mathfrak{x}, t)\, \mathfrak{v}(\mathfrak{x}, t)\right] dV.$$

Diese Beziehung gilt für jedes endliche reguläre Gebiet G und jede Zeit t. Aus der Stetigkeit des Integranden folgt daher

$$\varrho \frac{D\mathfrak{v}}{Dt} + a\,\varrho\,\mathfrak{v} + \nabla p = \mathfrak{K} \tag{1.13}$$

bzw. nach (1.8)

$$\varrho\left[\frac{\partial \mathfrak{v}}{\partial t} + (\mathfrak{v}\nabla)\,\mathfrak{v} + a\,\mathfrak{v}\right] + \nabla p = \mathfrak{K}. \tag{1.14}$$

Mit der Kontinuitätsgleichung (1.2) und den Gleichungen (1.14), die im reibungslosen Fall $a=0$ in die Eulerschen Gleichungen übergehen, haben wir vier Beziehungen zwischen $\mathfrak{v}$, p und ϱ gewonnen. Um eine weitere Beziehung zu erhalten, wollen wir die Energieverhältnisse diskutieren.

Die Energiedichte setzt sich aus dem kinetischen Energieanteil $\frac{1}{2}\varrho\,\mathfrak{v}^2$ und dem inneren Energieanteil $\varrho\,e$ zusammen. Die zur Zeit t in G enthaltene Gesamtenergie beträgt also

$$\int_G (\varrho\,e + \tfrac{1}{2}\varrho\,\mathfrak{v}^2)\,dV. \tag{1.15}$$

Mit der Flüssigkeitsströmung ist ein Energietransport der Stärke

$$\int_F (\varrho\,e + \tfrac{1}{2}\varrho\,\mathfrak{v}^2)\,\mathfrak{v}\,\mathfrak{n}\,dF \tag{1.16}$$

durch die Randfläche F von G verbunden. Die Energieproduktion in G beträgt also zur Zeit t

$$\frac{d}{dt}\int_G \left(\varrho\,e + \frac{1}{2}\varrho\,\mathfrak{v}^2\right)dV + \int_F \left(\varrho\,e + \frac{1}{2}\varrho\,\mathfrak{v}^2\right)\mathfrak{v}\,\mathfrak{n}\,dF. \tag{1.17}$$

Setzen wir voraus, daß der Flüssigkeit keine Wärme von außen zugeführt wird, und sehen wir von den Einflüssen der Wärmeleitung ab, so ist nach dem Energieprinzip die Energieproduktion in G gleich der Arbeit, die zur Zeit t durch die auf μ wirkende äußere Kraft (1.11) und die Druckkraft (1.9) geleistet wird. Die Leistungen der Kräfte (1.9) und (1.11) erhält man, indem man jeweils den Integranden skalar mit $\mathfrak{v}$ multipliziert. Somit gilt

$$\int_G \frac{\partial}{\partial t}\left(\varrho\,e + \frac{1}{2}\varrho\,\mathfrak{v}^2\right)dV + \int_F \left(\varrho\,e + \frac{1}{2}\varrho\,\mathfrak{v}^2\right)\mathfrak{v}\,\mathfrak{n}\,dF = \int_G \mathfrak{K}\,\mathfrak{v}\,dV - \int_F p\,\mathfrak{v}\,\mathfrak{n}\,dF. \tag{1.18}$$

Nach dem Satz von GAUSS folgt hieraus

$$\mathfrak{v}\,\mathfrak{K} - \nabla(p\,\mathfrak{v}) = \frac{\partial}{\partial t}\left[\varrho\left(e + \frac{1}{2}\mathfrak{v}^2\right)\right] + \nabla\left[\varrho\,\mathfrak{v}\left(e + \frac{1}{2}\mathfrak{v}^2\right)\right]. \tag{1.19}$$

Nach (1.2) und (1.7) gilt

$$\mathfrak{v}\,\mathfrak{K} - \nabla(p\,\mathfrak{v}) = \varrho\,\frac{D}{Dt}\left(e + \frac{1}{2}\mathfrak{v}^2\right). \tag{1.20}$$

Nach (1.13) ist

$$\varrho\,\frac{D}{Dt}\left(\frac{1}{2}\mathfrak{v}^2\right) = \varrho\,\mathfrak{v}\,\frac{D\mathfrak{v}}{Dt} = \mathfrak{v}\,\mathfrak{K} - \mathfrak{v}\,\nabla p - a\,\varrho\,\mathfrak{v}^2.$$

Hieraus und aus (1.20) folgt

$$-p\,\nabla\mathfrak{v} = \varrho\,\frac{De}{Dt} - a\,\varrho\,\mathfrak{v}^2. \tag{1.21}$$

Nach (1.2) gilt

$$\nabla\mathfrak{v} = -\frac{1}{\varrho}\left(\frac{\partial\varrho}{\partial t} + \mathfrak{v}\,\nabla\varrho\right) = -\frac{1}{\varrho}\,\frac{D\varrho}{Dt} = \varrho\,\frac{D}{Dt}\,\frac{1}{\varrho}\,.$$

Damit geht (1.21) in

$$\frac{De}{Dt} + p\,\frac{D}{Dt}\,\frac{1}{\varrho} = a\,\mathfrak{v}^2 \tag{1.22}$$

über. Es sei $s(\mathfrak{x}, t)$ die auf die Masseneinheit bezogene spezifische Entropie im Punkt $\mathfrak{x}$ zur Zeit t. Nach dem zweiten Hauptsatz der Thermodynamik kann die spezifische innere Energie $e(\mathfrak{x}, t)$ als Funktion $e(s, 1/\varrho)$ von s und $1/\varrho$ dargestellt werden, und es gilt

$$\frac{\partial e}{\partial s} = T, \qquad \frac{\partial e}{\partial\frac{1}{\varrho}} = -p, \tag{1.23}$$

wobei T die absolute Temperatur ist. Aus (1.23) folgt

$$\frac{De}{Dt} = T\,\frac{Ds}{Dt} - p\,\frac{D}{Dt}\,\frac{1}{\varrho}\,. \tag{1.24}$$

Hieraus und aus (1.22) ergibt sich die Beziehung

$$\frac{Ds}{Dt} = \frac{a\,\mathfrak{v}^2}{T}\,{}^{\star}. \tag{1.25}$$

Die Kontinuitätsgleichung (1.2), die Eulerschen Gleichungen (1.14) sowie die Energiegleichung (1.19) bzw. die zu ihr im wesentlichen äquivalente Entropiegleichung (1.25) ergeben zusammen fünf partielle Differentialgleichungen für die Komponenten von $\mathfrak{v}$ und zwei der thermodynamischen Größen p, ϱ, e, T, s. Die weiteren thermodynamischen Größen lassen sich mit Hilfe der Zustandsgleichung des betreffenden Mediums ermitteln.

Bei der Anwendung auf akustische Fragestellungen sind vor allem diejenigen Lösungen des Differentialgleichungssystems (1.2), (1.14), (1.25) von Bedeutung, für die die Geschwindigkeit $\mathfrak{v}$ und die Abweichungen $p_1 = p - p_0$ und $\varrho_1 = \varrho - \varrho_0$ von der Druck- und der Dichteverteilung im kräftefreien Ruhezustand klein sind. Hierdurch wird folgende Linearisierung nahegelegt [*3*]:

a) Die Zustandsgleichung $p = p(\varrho, s)$ des betreffenden Mediums kann durch die lineare Gleichung

$$p - p_0 = c^2(\varrho - \varrho_0) + A(s - s_0) \tag{1.26}$$

mit

$$c^2 = \left.\frac{\partial p}{\partial\varrho}\right|_{\substack{s=\text{const}\\ \varrho=\varrho_0,\, s=s_0}} \qquad A = \left.\frac{\partial p}{\partial s}\right|_{\substack{\varrho=\text{const}\\ \varrho=\varrho_0,\, s=s_0}}$$

ersetzt werden. Hierbei bedeutet $s_0(\mathfrak{x})$ die spezifische Entropie im Ruhezustand. Da der Druck p_0 im kräftefreien Gleichgewichtszustand konstant ist, folgt aus

$$\nabla p = \frac{\partial p}{\partial\varrho}\,\nabla\varrho + \frac{\partial p}{\partial s}\,\nabla s \quad \text{für} \quad \varrho = \varrho_0 \quad \text{und} \quad s = s_0$$

$$c^2\,\nabla\varrho_0 + A\,\nabla s_0 = 0. \tag{1.27}$$

★ Die zu (1.25) führende Argumentation ist für $a = 0$ in der Theorie der kompressiblen Strömung gebräuchlich. Vgl. etwa [*8*], S. 188ff.

b) Die Entropiegleichung

$$\frac{\partial s}{\partial t} + \mathfrak{v}\,\nabla s = \frac{a\,\mathfrak{v}^2}{T} \tag{1.28}$$

wird durch die lineare Gleichung

$$\frac{\partial s}{\partial t} + \mathfrak{v}\,\nabla s_0 = 0 \tag{1.29}$$

ersetzt. Eliminieren wir s mit Hilfe von (1.26), so geht (1.29) in

$$\frac{\partial p}{\partial t} - c^2\frac{\partial \varrho}{\partial t} + A\,\mathfrak{v}\,\nabla s_0 = 0 \tag{1.30}$$

über. Nach (1.27) folgt hieraus

$$\frac{\partial p}{\partial t} = c^2\left(\frac{\partial \varrho}{\partial t} + \mathfrak{v}\,\nabla \varrho_0\right). \tag{1.31}$$

c) Die Linearisierung der Kontinuitätsgleichung (1.2) ergibt

$$\frac{\partial \varrho}{\partial t} + \mathfrak{v}\,\nabla \varrho_0 + \varrho_0\,\nabla \mathfrak{v} = 0\,. \tag{1.32}$$

Nach (1.31) ist daher

$$\frac{\partial p}{\partial t} + \varrho_0\,c^2\,\nabla \mathfrak{v} = 0\,. \tag{1.33}$$

d) Aus den Eulerschen Gleichungen (1.14) folgt durch Linearisierung

$$\varrho_0\,\frac{\partial \mathfrak{v}}{\partial t} + a\,\varrho_0\,\mathfrak{v} + \nabla p = \mathfrak{K}\,. \tag{1.34}$$

Mit (1.33) und (1.34) haben wir vier lineare Differentialgleichungen zur Bestimmung von $\mathfrak{v}$ und p gewonnen. Zu ihrer weiteren Untersuchung setzen wir voraus, daß die Kraftdichteverteilung $\mathfrak{K}$ ein Potential Ψ besitzt. Es sei also

$$\mathfrak{K} = \nabla_{\mathfrak{x}}\,\Psi. \tag{1.35}$$

Aus (1.34) und (1.35) folgt

$$\frac{\partial}{\partial t}\left[\nabla\times(\varrho_0\,\mathfrak{v})\right] + a\,\nabla\times(\varrho_0\,\mathfrak{v}) = 0\,. \tag{1.36}$$

Es gibt also ein nicht von der Zeit abhängendes Vektorfeld $\mathfrak{c}(\mathfrak{x})$ mit

$$\nabla\times(\varrho_0\,\mathfrak{v}) = \mathfrak{c}\,e^{-a t}. \tag{1.37}$$

Aus (1.37) ersieht man: Verschwindet $\nabla\times(\varrho_0\,\mathfrak{v})$ zu einer Zeit t_0 im Punkt $\mathfrak{x}$, so gilt für jede Zeit t im Punkt $\mathfrak{x}$ $\nabla\times(\varrho_0\,\mathfrak{v}) = 0$. In der Theorie der stationären akustischen Wellenfelder beschränkt man sich auf die Untersuchung derjenigen Lösungen $\mathfrak{v}$, die in jedem Punkt $\mathfrak{x}$ zeitharmonisch sind. Wir können daher annehmen, daß für alle $\mathfrak{x}$ und t

$$\nabla\times(\varrho_0\,\mathfrak{v}) = 0 \tag{1.38}$$

gilt. Wegen (1.38) gibt es eine Funktion $\Phi(\mathfrak{x}, t)$ mit

$$\varrho_0\,\mathfrak{v} = \nabla_{\mathfrak{x}}\,\Phi. \tag{1.39}$$

Wir zeigen, daß Φ so gewählt werden kann, daß

(1.40) $$\frac{\partial \Phi}{\partial t} = \Psi - (p - p_0) - a\,\Phi$$

gilt. Hierzu betrachten wir mit festem $\mathfrak{x}_0$ die spezielle Lösung $\Phi_0 = \int_{\mathfrak{x}_0}^{\mathfrak{x}} \varrho_0\, \mathfrak{w}\, \mathfrak{t}\, ds$ von (1.39). Mit $\mathfrak{w}$ ist auch Φ_0 stetig nach t differenzierbar. Aus (1.34) und (1.35) folgt

(1.41) $$\nabla\left(\frac{\partial \Phi_0}{\partial t}\right) = \nabla\left[\Psi - (p - p_0) - a\,\Phi_0\right].$$

Es gibt daher eine nur von t abhängige Funktion $f(t)$ mit

(1.42) $$\frac{\partial \Phi_0}{\partial t} = \Psi - (p - p_0) - a\,\Phi_0 + f.$$

Mit Φ_0 ist auch

(1.43) $$\Phi = \Phi_0 + c(t)$$

eine Lösung von (1.39), wobei $c(t)$ eine beliebige zeitabhängige Funktion ist. Ist $c(t)$ speziell eine Lösung der Differentialgleichung

(1.44) $$c'(t) = -\,a\,c(t) + f(t),$$

so folgt aus (1.42), daß (1.40) durch $\Phi_0 + c(t)$ erfüllt wird. Φ ist durch (1.39) und (1.40) bis auf ein additives Glied der Form $c\,e^{-at}$ eindeutig bestimmt.

Aus (1.33), (1.39) und (1.40) folgt, daß Φ der Gleichung

(1.45) $$\varrho_0 \nabla\left(\frac{1}{\varrho_0}\nabla \Phi\right) = \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} + \frac{a}{c^2}\frac{\partial \Phi}{\partial t} - \frac{1}{c^2}\frac{\partial \Psi}{\partial t}$$

genügt. Ist umgekehrt Φ eine Lösung von (1.45), so erfüllen

(1.46) $$\mathfrak{v} = \frac{1}{\varrho_0}\nabla \Phi$$

und

(1.47) $$p - p_0 = \Psi - \frac{\partial \Phi}{\partial t} - a\,\Phi$$

die Gleichungen (1.33) und (1.34) mit $\mathfrak{K} = \nabla \Psi$. Für den Fall, daß die Kraftdichteverteilung $\mathfrak{K}$ ein Potential Ψ besitzt, ist damit die Untersuchung akustischer Wellenfelder im Rahmen der vorstehend entwickelten linearisierten Theorie auf die Diskussion der linearen partiellen Differentialgleichung (1.45) zurückgeführt. Die für die Schallverteilung charakteristischen Größen $\mathfrak{v}$ und p lassen sich aus den Lösungen Φ von (1.45) mit Hilfe von (1.46) und (1.47) berechnen. Zwischen p, s und ϱ besteht in der linearisierten Theorie der Zusammenhang (1.26). Speziell gilt für $s = s_0$

(1.48) $$p(\varrho, s_0) - p_0 = c^2(\varrho - \varrho_0).$$

Ist ϱ_0 vom Ort abhängig, so hängt auch die durch

(1.49) $$c^2 = \left.\frac{\partial p}{\partial \varrho}\right|_{\substack{s=\mathrm{const} \\ \varrho=\varrho_0,\, s=s_0}}$$

definierte Größe c vom Ort ab. Im Fall eines homogenen Mediums kann c als Schallgeschwindigkeit interpretiert werden, wie in § 3 noch ausgeführt wird.

Wir lassen jetzt die Voraussetzung fallen, daß das betrachtete Medium unbegrenzt ist, und nehmen an, daß Grenzflächen vorhanden sind, die das gasförmige oder flüssige Medium M_1 mit der mittleren Dichte ϱ_{01}, der Schallgeschwindigkeit c_1 und der Reibungskonstanten a_1 von einem anderen Medium M_2 mit den entsprechenden Größen ϱ_{02}, c_2, a_2 oder von einem festen Körper K trennen. An der Grenzfläche $F(t)$ zwischen M_1 und M_2 müssen die Drucke und die Normalkomponenten der Geschwindigkeit in beiden Medien gleich sein. Im Anwendungsbereich der linearisierten Theorie vollführen die einzelnen Flüssigkeitspunkte kleine Schwingungen um eine feste Ruhelage. Es ist daher möglich, die momentane Grenzfläche $F(t)$ durch die Grenzfläche F zwischen M_1 und M_2 im Ruhezustand zu ersetzen. Damit erhalten wir nach (1.46) und (1.47) für $\mathfrak{x}\in F$ die Grenzbedingungen

$$\frac{1}{\varrho_{01}}\frac{\partial\Phi_1}{\partial n}=\frac{1}{\varrho_{02}}\frac{\partial\Phi_2}{\partial n} \tag{1.50}$$

und

$$\frac{\partial\Phi_1}{\partial t}+a_1\Phi_1-\Psi_1=\frac{\partial\Phi_2}{\partial t}+a_2\Phi_2-\Psi_2. \tag{1.51}$$

Hierbei bedeutet, falls wir den in das Medium M_1 weisenden Normalenvektor von F im Punkt $\mathfrak{x}$ mit $\mathfrak{n}(\mathfrak{x})$ bezeichnen,

$$\begin{aligned}\Phi_1(\mathfrak{x},t)&=\lim_{\lambda\to+0}\Phi\big(\mathfrak{x}+\lambda\,\mathfrak{n}(\mathfrak{x}),t\big)\\ \Phi_2(\mathfrak{x},t)&=\lim_{\lambda\to+0}\Phi\big(\mathfrak{x}-\lambda\,\mathfrak{n}(\mathfrak{x}),t\big)\\ \frac{\partial}{\partial n}\Phi_1(\mathfrak{x},t)&=\lim_{\lambda\to+0}\frac{\partial}{\partial\mathfrak{n}(\mathfrak{x})}\Phi\big(\mathfrak{x}+\lambda\,\mathfrak{n}(\mathfrak{x}),t\big)\\ \frac{\partial}{\partial n}\Phi_2(\mathfrak{x},t)&=\lim_{\lambda\to+0}\frac{\partial}{\partial\mathfrak{n}(\mathfrak{x})}\Phi\big(\mathfrak{x}-\lambda\,\mathfrak{n}(\mathfrak{x}),t\big).\end{aligned} \tag{1.52}$$

Entsprechend können Grenzbedingungen an den Grenzflächen $F'(t)$ zwischen M_1 und K aufgestellt werden. Wir betrachten den Fall, daß der Körper K elastische Schwingungen um eine feste Ruhelage vollführt. Dann wird dem Medium M_1 an der jeweiligen Grenzfläche $F'(t)$ die Normalgeschwindigkeit $v_n(\mathfrak{x},t)$ des schwingenden Körpers K aufgezwungen. Nehmen wir an, daß die maximale Auslenkung der Randpunkte von K aus ihrer Ruhelage klein ist, so kann $F'(t)$ durch die Randfläche F' von K im ruhenden Zustand ersetzt werden. Damit erhalten wir nach (1.46) für $\mathfrak{x}\in F'$ die Randbedingung

$$\frac{\partial}{\partial n}\Phi(\mathfrak{x},t)=g(\mathfrak{x},t), \tag{1.53}$$

wobei $g(\mathfrak{x},t)$ eine gegebene auf F' erklärte Funktion ist.

Die auf die Masseneinheit bezogene Energiedichte eines akustischen Wellenfeldes beträgt

$$e+\tfrac{1}{2}\mathfrak{v}^2. \tag{1.54}$$

Setzen wir $1/\varrho=v$, so berechnet sich e nach (1.23) durch

$$e(\mathfrak{x},t)=-\int\limits_{\frac{1}{\varrho_0(\mathfrak{x})}}^{\frac{1}{\varrho(\mathfrak{x},t)}}p\big(v,s_0(\mathfrak{x})\big)\,dv+\int\limits_{s_0(\mathfrak{x})}^{s(\mathfrak{x},t)}T\Big(\frac{1}{\varrho(\mathfrak{x},t)},s\Big)\,ds+e_0(\mathfrak{x}), \tag{1.55}$$

wobei $e_0(\mathfrak{x})$ die Verteilung der inneren Energie im Ruhezustand und $p=p(v,s)$ die Zustandsgleichung des betreffenden Mediums ist. Die innere Energiedichte e kann in die Anteile

$$e_0-p_0\left(\frac{1}{\varrho}-\frac{1}{\varrho_0}\right)+\int\limits_{s_0}^{s} T\left(\frac{1}{\varrho},s\right)ds \qquad \text{und} \qquad -\int\limits_{\frac{1}{\varrho_0}}^{\frac{1}{\varrho}}(p-p_0)\,dv$$

aufgespalten werden. In der Akustik interessiert vor allem der zweite Anteil, der nach RAYLEIGH [*12*] als potentielle Energiedichte ε_{pot} angesehen werden kann. Wir setzen also

$$\varepsilon_{\text{pot}}(\mathfrak{x},t)=-\int\limits_{\frac{1}{\varrho_0(\mathfrak{x})}}^{\frac{1}{\varrho(\mathfrak{x},t)}}\left[p(v,s_0(\mathfrak{x}))-p_0\right]dv. \tag{1.56}$$

Die potentielle Energie

$$\int\limits_G \varrho\,\varepsilon_{\text{pot}}\,dV$$

eines Flüssigkeitsteilchen μ, das zur Zeit t das Gebiet G ausfüllt, kann als die Arbeit interpretiert werden, die der Überdruck $p-p_0$ leistet, um μ aus dem Ruhezustand in den herrschenden physikalischen Zustand zu überführen.

Wir wollen ε_{pot} im Rahmen der linearisierten Theorie durch Φ ausdrücken. Hierbei können wir annehmen, daß zwischen p und ϱ die lineare Beziehung (1.48) besteht. Setzen wir $\frac{1}{\varrho_0}(\varrho_1-\varrho_0)=\sigma_1$, so gilt für $\sigma_1\to 0$

$$-\int\limits_{\frac{1}{\varrho_0}}^{\frac{1}{\varrho_1}}(p-p_0)\,dv=-c^2\int\limits_{\varrho_0}^{\varrho_1}(\varrho-\varrho_0)\left(-\frac{1}{\varrho^2}\right)d\varrho=c^2\left(\ln\frac{\varrho_1}{\varrho_0}+\frac{\varrho_0}{\varrho_1}-1\right)$$
$$=c^2\left[\ln(1+\sigma_1)+\frac{1}{1+\sigma_1}-1\right]=c^2\frac{\sigma_1^2}{2}+O(\sigma_1^3).$$

Aus $\sigma_1=\frac{1}{\varrho_0}(\varrho_1-\varrho_0)=\frac{1}{\varrho_0 c^2}\left[p(\varrho_1,s_0)-p_0\right]$ folgt daher nach (1.47)

$$\varepsilon_{\text{pot}}=\frac{1}{2\varrho_0^2c^2}\left(\frac{\partial\Phi}{\partial t}+a\,\Phi-\Psi\right)^{2\star}. \tag{1.57}$$

Zur Unterscheidung von der Gesamtenergiedichte (1.54) wollen wir im folgenden die Summe der potentiellen Energiedichte (1.56) und der kinetischen Energiedichte $\frac{1}{2}\mathfrak{v}^2$ als akustische Energiedichte ε bezeichnen. In der linearisierten Theorie gilt nach (1.46) und (1.57)

$$\varepsilon=\frac{1}{2\varrho_0^2}\left[(\nabla\Phi)^2+\frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}+a\,\Phi-\Psi\right)^2\right]. \tag{1.58}$$

★ Hierbei wird angenommen, daß $s(\mathfrak{x},t)-s_0(\mathfrak{x})$ und damit auch

$$p(\varrho_1,s(\mathfrak{x},t))-p(\varrho_1,s_0(\mathfrak{x}))=p(\mathfrak{x},t)-p(\varrho_1,s_0)$$

die Größenordnung $O(\sigma_1^2)$ besitzt. Diese Annahme kann bei stationären Wellenfeldern genügend hoher Frequenz mit Hilfe von (1.28) bzw. (1.29) begründet werden.

Aus (1.45) und (1.58) folgt

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{\varrho_0^2}\left[\nabla\Phi\nabla\Phi_t + \frac{1}{c^2}(\Phi_t + a\Phi - \Psi)(\Phi_{tt} + a\Phi_t - \Psi_t)\right]$$

$$= \frac{1}{\varrho_0^2}\left[\nabla\Phi\nabla(\Phi_t + a\Phi - \Psi) + \varrho_0\nabla\left(\frac{1}{\varrho_0}\nabla\Phi\right)(\Phi_t + a\Phi - \Psi) - a(\nabla\Phi)^2 + \nabla\Phi\nabla\Psi\right],$$

also

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{\varrho_0}\nabla\left[\frac{1}{\varrho_0}(\Phi_t + a\Phi - \Psi)\nabla\Phi\right] - \frac{a}{\varrho_0^2}(\nabla\Phi)^2 + \frac{1}{\varrho_0^2}\nabla\Phi\nabla\Psi. \tag{1.59}$$

Es sei G ein endliches reguläres Gebiet mit der Randfläche F. Für die zur Zeit t im Inneren von G befindliche akustische Energie $E_G(t)$ gilt nach (1.59)

$$\begin{aligned}\frac{d}{dt}E_G(t) &= \int_G \varrho_0(\mathfrak{x})\frac{\partial}{\partial t}\varepsilon(\mathfrak{x},t)\,dV \\ &= \int_F \frac{1}{\varrho_0}(\Phi_t + a\Phi - \Psi)\frac{\partial\Phi}{\partial n}\,dF - a\int_G \frac{1}{\varrho_0}(\nabla\Phi)^2\,dV + \int_G \frac{1}{\varrho_0}\nabla\Phi\nabla\Psi\,dV.\end{aligned} \tag{1.60}$$

Im Fall, daß durch F keine Flüssigkeit transportiert wird, folgt aus (1.60) wegen $\partial\Phi/\partial n = 0$

$$\frac{d}{dt}E_G(t) = -a\int_G \frac{1}{\varrho_0}(\nabla\Phi)^2\,dV + \int_G \frac{1}{\varrho_0}\nabla\Phi\nabla\Psi\,dV. \tag{1.61}$$

Treten also keine äußeren Kräfte und keine Reibungskräfte auf, so erweist sich $E_G(t)$ als konstant. Für $a \neq 0$ kann

$$-\frac{a}{\varrho_0}(\nabla\Phi)^2 \tag{1.62}$$

als spezifischer Energieverlust durch Reibung interpretiert werden. Entsprechend stellt

$$\frac{1}{\varrho_0}\nabla\Phi\nabla\Psi \tag{1.63}$$

für $\nabla\Psi \neq 0$ die spezifische Energieproduktion der äußeren Kräfte dar. Übertragen wir diese Deutung auf den Fall, daß $\partial\Phi/\partial n$ nicht für alle $\mathfrak{x}$ auf F verschwindet, so folgt aus (1.60), daß der in das Äußere von G gerichtete akustische Energiefluß durch

$$S_F(t) = \int_F \frac{1}{\varrho_0}(\Psi - \Phi_t - a\Phi)\frac{\partial\Phi}{\partial n}\,dF \tag{1.64}$$

dargestellt wird. Der Vektor

$$\mathfrak{S} = \frac{1}{\varrho_0}(\Psi - \Phi_t - a\Phi)\nabla\Phi \tag{1.65}$$

kann daher — entsprechend dem Poyntingschen Vektor bei elektromagnetischen Schwingungsvorgängen — als akustischer Energieströmungsvektor interpretiert werden. Er genügt nach (1.59) der Gleichung

$$\varrho_0\frac{\partial\varepsilon}{\partial t} + \nabla\mathfrak{S} = \frac{1}{\varrho_0}\nabla\Phi\nabla\Psi - \frac{a}{\varrho_0}(\nabla\Phi)^2. \tag{1.66}$$

§ 2. Stationäre akustische Schwingungsvorgänge

Im folgenden setzen wir voraus, daß die äußeren Kräfte und die an den Randflächen fester Körper vorgegebenen Werte für die Normalgeschwindigkeit $\mathfrak{v}\,\mathfrak{n}$ zeitharmonisch mit der Frequenz ω sind. Für das Kraftdichtepotential Ψ kann also

$$(2.1)\qquad -\frac{1}{c^2}\frac{\partial}{\partial t}\Psi(\mathfrak{x},t)=f_1(\mathfrak{x})\cos\omega t+f_2(\mathfrak{x})\sin\omega t=\mathrm{Re}\{f(\mathfrak{x})\,e^{-i\omega t}\}$$

mit $f=f_1+i f_2$ gesetzt werden. Hierdurch geht (1.45) in

$$(2.2)\qquad \varrho_0\nabla\Big(\frac{1}{\varrho_0}\nabla\Phi\Big)=\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}+\frac{a}{c^2}\frac{\partial\Phi}{\partial t}+\mathrm{Re}\{f\,e^{-i\omega t}\}$$

über.

In der vorliegenden Arbeit werden Lösungen von (2.2) untersucht, die die Gestalt

$$(2.3)\qquad \Phi(\mathfrak{x},t)=U_1(\mathfrak{x})\cos\omega t+U_2(\mathfrak{x})\sin\omega t=\mathrm{Re}\{U(\mathfrak{x})\,e^{-i\omega t}\}$$

mit $U=U_1+i\,U_2$ besitzen. Die Lösungen der Gestalt (2.3) charakterisieren die eingeschwungenen "stationären" akustischen Schwingungsvorgänge.

Nach (2.2) und (2.3) gilt für alle t

$$\begin{aligned}\varrho_0\nabla\Big(\frac{1}{\varrho_0}\nabla\Phi\Big)&-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}-\frac{a}{c^2}\frac{\partial\Phi}{\partial t}-\mathrm{Re}\{f\,e^{-i\omega t}\}\\&=\mathrm{Re}\Big\{\Big[\varrho_0\nabla\Big(\frac{1}{\varrho_0}\nabla U\Big)+\frac{\omega^2}{c^2}U+\frac{i\,a\,\omega}{c^2}U-f\Big]e^{-i\omega t}\Big\}\\&=\cos\omega t\cdot\mathrm{Re}\Big[\varrho_0\nabla\Big(\frac{1}{\varrho_0}\nabla U\Big)+\frac{\omega}{c^2}(\omega+i\,a)\,U-f\Big]+\\&\quad+\sin\omega t\cdot\mathrm{Im}\Big[\varrho_0\nabla\Big(\frac{1}{\varrho_0}\nabla U\Big)+\frac{\omega}{c^2}(\omega+i\,a)\,U-f\Big]=0.\end{aligned}$$

Für $t=0$ bzw. $t=\pi/2\omega$ folgt hieraus, daß Real- und Imaginärteil von

$$\varrho_0\nabla\Big(\frac{1}{\varrho_0}\nabla U\Big)+\frac{\omega}{c^2}(\omega+i\,a)\,U-f$$

verschwinden. Somit gilt

$$(2.4)\qquad \varrho_0\nabla\Big(\frac{1}{\varrho_0}\nabla U\Big)+\varkappa^2 U=f$$

mit

$$(2.5)\qquad \varkappa^2=\frac{\omega}{c^2}(\omega+i\,a).$$

Man erhält also sämtliche stationären Lösungen von (2.2), indem man alle Lösungen U von (2.4) bestimmt und

$$\Phi(\mathfrak{x},t)=\mathrm{Re}\{U(\mathfrak{x})\,e^{-i\omega t}\}$$

setzt. Da ω, a und c nicht negativ sind, gilt $0\leqq\mathrm{arc}(\varkappa^2)<\frac{1}{2}\pi$. Wir können daher $\varkappa$ so wählen, daß

$$(2.6)\qquad 0\leqq\mathrm{arc}\,\varkappa(\mathfrak{x})<\tfrac{1}{4}\pi$$

ist.

Beachten wir, daß die äußeren Kräfte wegen (2.1) das Kraftdichtepotential

$$\Psi(\mathfrak{x}, t) = -\frac{c^2}{\omega} \operatorname{Re}\{i f(\mathfrak{x})\, e^{-i\omega t}\} \tag{2.7}$$

besitzen, so folgt aus (1.46) und (1.47), daß die für das Schallfeld charakteristischen Größen $\mathfrak{v}$, p durch

$$\mathfrak{v} = \frac{1}{\varrho_0} \operatorname{Re}\{\nabla U\, e^{-i\omega t}\} \tag{2.8}$$

und

$$p - p_0 = -\operatorname{Re}\left\{\left[(a - i\omega)\, U + i\frac{c^2}{\omega} f\right] e^{-i\omega t}\right\} \tag{2.9}$$

dargestellt werden.

Bei dem Durchgang durch die Grenzfläche F zweier Medien M_1 und M_2 sind die Normalkomponente der Geschwindigkeit und der Druck stetig. Hieraus ergeben sich nach (2.8) und (2.9) für $\mathfrak{x} \in F$ die Bedingungen

$$\frac{1}{\varrho_{01}} \frac{\partial U_1}{\partial n} = \frac{1}{\varrho_{02}} \frac{\partial U_2}{\partial n} \tag{2.10}$$

und

$$(a_1 - i\omega)\, U_1 + i\frac{c_1^2}{\omega} f = (a_2 - i\omega)\, U_2 + i\frac{c_2^2}{\omega} f. \tag{2.11}$$

Hierbei bedeutet ϱ_{0i} die mittlere Dichte, c_i die Schallgeschwindigkeit und a_i die Reibungskonstante des Mediums M_i $(i = 1, 2)$, während U_1, $\partial U_1/\partial n, \ldots$ in der gleichen Weise wie Φ_1, $\partial\Phi_1/\partial n, \ldots$ in (1.52) erklärt werden können.

Für die Punkte der Randfläche eines festen Körpers erhalten wir nach (1.53), falls wir

$$g(\mathfrak{x}, t) = \operatorname{Re}\{g(\mathfrak{x})\, e^{-i\omega t}\} \tag{2.12}$$

setzen, die Randbedingung

$$\frac{\partial}{\partial n} U(\mathfrak{x}) = g(\mathfrak{x}). \tag{2.13}$$

Wir wollen jetzt die Energieverteilung bei stationären akustischen Schwingungsvorgängen diskutieren. Wir beschränken uns auf die Untersuchung von Raumteilen, in denen keine äußeren Kräfte wirken. Bei stationären Vorgängen interessieren vor allem die zeitlichen Mittel $\tilde{\varepsilon}(\mathfrak{x})$ und $\tilde{\mathfrak{S}}(\mathfrak{x})$ der akustischen Energiedichte $\varepsilon(\mathfrak{x}, t)$ und des akustischen Energiestromvektors $\mathfrak{S}(\mathfrak{x}, t)$:

$$\tilde{\varepsilon}(\mathfrak{x}) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \varepsilon(\mathfrak{x}, t)\, dt \tag{2.14}$$

und

$$\tilde{\mathfrak{S}}(\mathfrak{x}) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \mathfrak{S}(\mathfrak{x}, t)\, dt. \tag{2.15}$$

Nach (1.58) und (2.3) ist

$$\begin{aligned}
\varepsilon(\mathfrak{x}, t) &= \frac{1}{2\varrho_0^2}\left[(\nabla\Phi)^2 + \frac{1}{c^2}(\Phi_t + a\,\Phi)^2\right] \\
&= \frac{1}{2\varrho_0^2}\left\{[\operatorname{Re}(\nabla U\, e^{-i\omega t})]^2 + \frac{1}{c^2}[\operatorname{Re}((a - i\omega)\, U e^{-i\omega t})]^2\right\}.
\end{aligned}$$

Aus

$$[\mathrm{Re}(A\,e^{-i\omega t})]^2 = \tfrac{1}{4}(A\,e^{-i\omega t} + \bar{A}\,e^{i\omega t})^2 = \tfrac{1}{4}(A^2 e^{-2i\omega t} + \bar{A}^2 e^{2i\omega t}) + \tfrac{1}{2}|A|^2$$

folgt

$$\tilde{\varepsilon}(\mathfrak{x}) = \frac{1}{4\varrho_0^2}\left[|\nabla U|^2 + \frac{\omega^2 + a^2}{c^2}|U|^2\right]. \tag{2.16}$$

Im Fall $a = 0$ spezialisiert sich (2.16) wegen $\varkappa = \omega/c$ zu

$$\tilde{\varepsilon}(\mathfrak{x}) = \frac{1}{4\varrho_0^2}\left[|\nabla U|^2 + \varkappa^2 |U|^2\right]. \tag{2.17}$$

Entsprechend erhält man wegen

$$\begin{aligned}\mathfrak{S}(\mathfrak{x}, t) &= -\frac{1}{\varrho_0}\left(\frac{\partial \Phi}{\partial t} + a\,\Phi\right)\nabla\Phi \\ &= -\frac{1}{\varrho_0}\left[\mathrm{Re}\left((a - i\omega)\,U\,e^{-i\omega t}\right)\right]\cdot\left[\mathrm{Re}(\nabla U\,e^{-i\omega t})\right]\end{aligned}$$

und

$$\mathrm{Re}(A\,e^{-i\omega t})\cdot\mathrm{Re}(B\,e^{-i\omega t}) = \tfrac{1}{4}[A\,B\,e^{-2i\omega t} + \bar{A}\,\bar{B}\,e^{2i\omega t} + A\,\bar{B} + \bar{A}\,B]$$

$$\begin{aligned}\tilde{\mathfrak{S}}(\mathfrak{x}) &= -\frac{1}{4\varrho_0}\{(a - i\omega)\,U\nabla\bar{U} + (a + i\omega)\,\bar{U}\nabla U\} \\ &= -\frac{1}{4\varrho_0}\{a(U\nabla\bar{U} + \bar{U}\nabla U) - i\omega(U\nabla\bar{U} - \bar{U}\nabla U)\} \\ &= \frac{1}{2\varrho_0}\{\omega\,\mathrm{Im}(\bar{U}\nabla U) - a\,\mathrm{Re}(\bar{U}\nabla U)\},\end{aligned}$$

also

$$\tilde{\mathfrak{S}}(\mathfrak{x}) = \frac{1}{2\varrho_0}\mathrm{Im}\{(\omega - i\,a)\,\bar{U}\nabla U\}. \tag{2.18}$$

Im Fall $a = 0$ spezialisiert sich (2.18) zu

$$\tilde{\mathfrak{S}}(\mathfrak{x}) = \frac{\omega}{2\varrho_0}\mathrm{Im}(\bar{U}\nabla U). \tag{2.19}$$

Das zeitliche Mittel $\tilde{S}_F$ des Energieflusses $S_F(t)$ durch die geschlossene Fläche F berechnet sich nach (1.64) und (2.18) zu

$$\tilde{S}_F = \frac{1}{2}\mathrm{Im}\left\{(\omega - i\,a)\int_F \frac{1}{\varrho_0}\bar{U}\frac{\partial U}{\partial n}\,dF\right\}. \tag{2.20}$$

Damit haben wir die für Energiebetrachtungen bei stationären akustischen Vorgängen wesentlichen Größen $\tilde{\varepsilon}$, $\tilde{\mathfrak{S}}$ und $\tilde{S}_F$ durch U ausgedrückt.

§ 3. Grundlösungen der Schwingungsgleichung

Im folgenden setzen wir voraus, daß für die von uns betrachteten Medien die Dichte ϱ_0 im Ruhezustand und die Schallgeschwindigkeit c für $|\mathfrak{x}| > R$ konstant ist und daß die Kraftdichteverteilung $\mathfrak{f}$ für $|\mathfrak{x}| > R$ verschwindet. Für $|\mathfrak{x}| > R$ geht dann die Gleichung

$$\varrho_0\nabla\left(\frac{1}{\varrho_0}\nabla U\right) + \varkappa^2 U = \mathfrak{f} \tag{3.1}$$

in die Helmholtzsche Schwingungsgleichung

$$\Delta U + \varkappa^2 U = 0 \tag{3.2}$$

mit konstantem $\varkappa$ über. In diesem Abschnitt sollen einige spezielle Lösungen von (3.2) untersucht werden, aus denen wir später die Lösungen der in der Einleitung formulierten Probleme aufbauen werden.

Wir fragen zunächst nach Lösungen von (3.2), die nur von dem Abstand $|\mathfrak{x}-\mathfrak{x}'|=r$ von einem festen Punkt $\mathfrak{x}'$ abhängen. Für $U(\mathfrak{x})=y(r)$ erhalten wir die Gleichung

$$y''+\frac{2}{r}y'+\varkappa^2 y=0 \tag{3.3}$$

mit der allgemeinen Lösung

$$y(r)=A\,\frac{e^{i\varkappa r}}{r}+B\,\frac{e^{-i\varkappa r}}{r}\,. \tag{3.4}$$

Der Lösung $e^{i\varkappa r}/r$ von (3.2) entspricht nach (2.3) die Lösung

$$\Phi(\mathfrak{x},t)=\mathrm{Re}\left\{\frac{e^{i(\varkappa r-\omega t)}}{r}\right\}=\frac{1}{r}\,e^{-\varkappa_2 r}\cos(\varkappa_1 r-\omega t) \tag{3.5}$$

der homogenen Wellengleichung

$$\Delta\Phi=\frac{1}{c^2}\,\frac{\partial^2\Phi}{\partial t^2}+\frac{a}{c^2}\,\frac{\partial\Phi}{\partial t}\,. \tag{3.6}$$

Wir hatten $\varkappa$ so gewählt, daß $\varkappa_1=\mathrm{Re}(\varkappa)>0$ und $\varkappa_2=\mathrm{Im}(\varkappa)\geqq 0$ ist. Die Funktion (3.5) beschreibt daher einen vom Punkt $\mathfrak{x}'$ ausgehenden Wellenvorgang mit der Amplitude $\frac{1}{r}\,e^{-\varkappa_2 r}$.

Die Flächen konstanter Phase $\varkappa_1 r-\omega t$ sind die Kugeln $r=\frac{\omega}{\varkappa_1}\,t+\mathrm{const}$ mit dem Mittelpunkt $\mathfrak{x}'$. Ihre Radien vergrößern sich für wachsendes t mit der konstanten Geschwindigkeit

$$c_1=\frac{\omega}{\varkappa_1}\,. \tag{3.7}$$

Der durch (3.5) beschriebene Schwingungsvorgang stellt ein stationäres Wellenfeld dar, das durch eine im Punkt $\mathfrak{x}'$ wirkende periodische Störung mit der Frequenz ω erzeugt wird. Für $a=0$ folgt aus (3.7) $c_1=c$, so daß die in (1.49) eingeführte Größe c als Schallgeschwindigkeit des betreffenden Mediums interpretiert werden kann. Für $a\neq 0$ gilt

$$(\omega+i\,a)^{\frac{1}{2}}=\pm\frac{1}{\sqrt{2}}\left\{[(\omega^2+a^2)^{\frac{1}{2}}+\omega]^{\frac{1}{2}}+i\,[(\omega^2+a^2)^{\frac{1}{2}}-\omega]^{\frac{1}{2}}\right\},$$

also wegen (2.5)

$$\varkappa_1=\frac{1}{c\sqrt{2}}\,[\omega(\omega^2+a^2)^{\frac{1}{2}}+\omega^2]^{\frac{1}{2}}=\frac{1}{c\sqrt{2}}\,\frac{a\,\omega}{[\omega(\omega^2+a^2)^{\frac{1}{2}}-\omega^2]^{\frac{1}{2}}} \tag{3.8}$$

und somit

$$c_1=c\,\frac{\omega}{a}\sqrt{2}\left[\left(1+\left(\frac{a}{\omega}\right)^2\right)^{\frac{1}{2}}-1\right]^{\frac{1}{2}}. \tag{3.9}$$

Es gilt also $c_1<c$. Der Quotient c_1/c hängt nur von a/ω ab und strebt für $a/\omega\to 0$ gegen 1, wie sich durch Reihenentwicklung ergibt. Bei auftretender Dämpfung erweist sich somit c als asymptotischer Wert der Phasengeschwindigkeit c_1 für hohe Frequenzen.

Zur Untersuchung des mit dem Wellenfeld (3.5) verbundenen Energietransportes berechnen wir den mittleren akustischen Energiefluß durch die Kugel Σ_R um $\mathfrak{x}'$ mit dem Radius R. Mit $U(\mathfrak{x}) = \frac{1}{r} e^{i\varkappa r}$ erhalten wir nach (2.20) und (2.5)

$$\tilde{S}_{\Sigma_R} = \frac{1}{2\varrho_0} \operatorname{Im}\left\{(\omega - i\,a) \int\limits_{|\mathfrak{x}-\mathfrak{x}'|=R} \overline{U} \frac{\partial U}{\partial r}\, dF_{\mathfrak{x}}\right\}$$

$$= \frac{2\pi}{\varrho_0} e^{-2\varkappa_2 R} \operatorname{Im}\left\{(\omega - i\,a)\left(i\varkappa - \frac{1}{R}\right)\right\},$$

also

(3.10) $$\tilde{S}_{\Sigma_R} = \frac{2\pi}{\varrho_0} e^{-2\varkappa_2 R}\left(\omega\varkappa_1 + a\varkappa_2 + \frac{a}{R}\right).$$

Der mittlere akustische Energiefluß durch Σ_R ist also für alle R positiv, so daß der Energietransport zum Unendlichen hin gerichtet ist. Im Fall $a = 0$ ist

(3.11) $$\tilde{S}_{\Sigma_R} = \frac{2\pi}{\varrho_0}\omega\varkappa = \frac{2\pi\omega^2}{\varrho_0 c}.$$

$\tilde{S}_{\Sigma_R}$ ist also von R unabhängig; der Punkt $\mathfrak{x}'$ ist eine akustische Energiequelle der mittleren Stärke $\frac{2\pi\omega^2}{\varrho_0 c}$.

Entsprechend wird durch $\frac{1}{r} e^{-i\varkappa r}$ bzw.

(3.12) $$\Phi(\mathfrak{x}, t) = \operatorname{Re}\left\{\frac{e^{-i(\varkappa r + \omega t)}}{r}\right\} = \frac{1}{r} e^{\varkappa_2 r} \cos(\varkappa_1 r + \omega t)$$

ein Wellenvorgang beschrieben, bei dem sich die Kugeln konstanter Phase $\varkappa_1 r + \omega t$ für wachsendes t mit der radialen Geschwindigkeit $-\frac{\omega}{\varkappa_1}$ auf den Punkt $\mathfrak{x}'$ zusammenziehen. Der mittlere akustische Energiefluß durch die Kugel Σ_R beträgt

(3.13) $$\tilde{S}_{\Sigma_R} = \frac{2\pi}{\varrho_0} e^{2\varkappa_2 R}\left(-\omega\varkappa_1 - a\varkappa_2 + \frac{a}{R}\right).$$

Es gibt also ein R_0, so daß $\tilde{S}_{\Sigma_R}$ für alle $R > R_0$ negativ ist. Das bedeutet, daß bei dem Schwingungsvorgang (3.12) fortwährend akustische Energie aus dem Unendlichen eingestrahlt wird. Ein solcher Vorgang läßt sich jedoch physikalisch durch Anordnungen, die im Endlichen getroffen werden, nicht realisieren.

Hiermit werden wir auf eine charakteristische, von SOMMERFELD [*15*] erkannte Schwierigkeit geführt, die im wesentlichen darauf beruht, daß physikalisch nur diejenigen akustischen Schwingungen sinnvoll sind, die sich im Unendlichen wie auslaufende (divergente) Wellenvorgänge verhalten, während die Schwingungsgleichung (3.2) auch die im mathematischen Sinn gleichwertigen, physikalisch jedoch nicht zu realisierenden vom Unendlichen her einstrahlenden (konvergenten) Wellenvorgänge umfaßt. Es ergibt sich daher die Notwendigkeit, die physikalisch realisierbaren Lösungen von (3.2) durch eine zusätzliche, sich auf das Verhalten im Unendlichen beziehende Bedingung zu kennzeichnen und diejenigen Lösungen auszuschließen, die konvergenten Wellen entsprechen oder durch Überlagerung von konvergenten und divergenten Wellen entstehen.

Nach SOMMERFELD kann man sich hierzu folgender Bedingung bedienen:

Sommerfeldsche Ausstrahlungsbedingung. Die Lösung $U(\mathfrak{x})$ von (3.2) erfülle für $|\mathfrak{x}| = r \to \infty$ und $|\mathfrak{x}_0| = 1$ die asymptotischen Relationen

$$U(r\,\mathfrak{x}_0) = O\left(\frac{1}{r}\right) \tag{3.14}$$

und

$$\frac{\partial}{\partial r} U(r\,\mathfrak{x}_0) - i\varkappa\, U(r\,\mathfrak{x}_0) = o\left(\frac{1}{r}\right) \tag{3.15}$$

gleichmäßig für alle Richtungen $\mathfrak{x}_0$.

Von den rotationssymmetrischen Lösungen (3.4) der Schwingungsgleichung erfüllen für $\mathfrak{x}' = 0$ genau die Funktionen der Form $A\,\frac{e^{i\varkappa r}}{r}$ die Ausstrahlungsbedingung, also gerade diejenigen Lösungen, die Ausstrahlungsvorgänge beschreiben.

Wir zeigen jetzt, daß die Funktion

$$U(\mathfrak{x}) = \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|} \tag{3.16}$$

auch für $\mathfrak{x}' \neq 0$ die Ausstrahlungsbedingung erfüllt. Bei festem $\mathfrak{x}'$ strebt mit $|\mathfrak{x}|$ auch $|\mathfrak{x}-\mathfrak{x}'|$ gegen ∞, so daß (3.14) gilt. Aus

$$\nabla_{\mathfrak{x}} \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|} = e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}\left(\frac{i\varkappa}{|\mathfrak{x}-\mathfrak{x}'|^2} - \frac{1}{|\mathfrak{x}-\mathfrak{x}'|^3}\right)(\mathfrak{x}-\mathfrak{x}') \tag{3.17}$$

folgt für $r \to \infty$

$$\nabla U(r\,\mathfrak{x}_0) = i\varkappa\, U(r\,\mathfrak{x}_0)\,\frac{\mathfrak{x}-\mathfrak{x}'}{|\mathfrak{x}-\mathfrak{x}'|} + O\left(\frac{1}{r^2}\right). \tag{3.18}$$

Wegen

$$\frac{\mathfrak{x}-\mathfrak{x}'}{|\mathfrak{x}-\mathfrak{x}'|} = \mathfrak{x}_0 + O\left(\frac{1}{r}\right) \tag{3.19}$$

und (3.14) gilt daher für $r \to \infty$

$$\nabla U(r\,\mathfrak{x}_0) = i\varkappa\, U(r\,\mathfrak{x}_0)\,\mathfrak{x}_0 + O\left(\frac{1}{r^2}\right). \tag{3.20}$$

Wegen $\partial/\partial r = \mathfrak{x}_0 \nabla$ erhält man hieraus (3.15) durch skalare Multiplikation mit $\mathfrak{x}_0$. Für $|\mathfrak{x}'| < C$ gelten die angegebenen Abschätzungen gleichmäßig in $\mathfrak{x}_0$ und $\mathfrak{x}'$.

Weitere Lösungen der Schwingungsgleichung (3.2), die der Ausstrahlungsbedingung genügen, kann man aus den Lösungen (3.16) durch Superposition gewinnen. Es entstehen Lösungen der Form

$$\sum_{j=1}^{n} A_j \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}_j|}}{|\mathfrak{x}-\mathfrak{x}_j|}. \tag{3.21}$$

Unter ihnen befinden sich die Lösungen

$$U_\tau(\mathfrak{x}) = \frac{1}{\tau}\left(\frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'-\tau\mathfrak{n}|}}{|\mathfrak{x}-\mathfrak{x}'-\tau\mathfrak{n}|} - \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|}\right). \tag{3.22}$$

Hierbei sei $\mathfrak{n}$ ein Einheitsvektor. Durch $U_\tau(\mathfrak{x})$ bzw.

$$\Phi_\tau(\mathfrak{x}, t) = \mathrm{Re}\{U_\tau(\mathfrak{x}) \cdot e^{-i\omega t}\} \tag{3.23}$$

wird ein Wellenfeld beschrieben, das durch zwei periodische Störungen gleicher Stärke, aber mit um π gegeneinander verschobenen Phasen erzeugt wird. Für $\tau \to 0$ erhält man die Funktion

$$U_0(\mathfrak{x}) = \frac{\partial}{\partial \mathfrak{n}_{\mathfrak{x}'}} \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|}, \tag{3.24}$$

die für $\mathfrak{x} \neq \mathfrak{x}'$ wegen der Vertauschbarkeit der partiellen Ableitungen ebenfalls eine Lösung von (3.2) ist. $U_0(\mathfrak{x})$ kann als Doppelquelle im Punkt $\mathfrak{x}'$ mit der Achse $\mathfrak{n}$ interpretiert werden. Wir zeigen, daß auch $U_0(\mathfrak{x})$ die Ausstrahlungsbedingung erfüllt. Hierzu wählen wir ein Koordinatensystem, dessen x-Achse die gleiche Richtung wie $\mathfrak{n}$ hat. Dann gilt mit $\mathfrak{x} = \{x, y, z\}$ und $\mathfrak{x}' = \{x', y', z'\}$

$$\begin{aligned} U_0(\mathfrak{x}) = \frac{\partial}{\partial x'} \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|} &= -\, e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}\left(\frac{i\varkappa}{|\mathfrak{x}-\mathfrak{x}'|^2} - \frac{1}{|\mathfrak{x}-\mathfrak{x}'|^3}\right)(x-x') \\ &= -\, i\varkappa\, e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|} \frac{x-x'}{|\mathfrak{x}-\mathfrak{x}'|^2} + O\left(\frac{1}{r^2}\right). \end{aligned} \tag{3.25}$$

Ferner folgt aus (3.17) durch Differentiation für $r \to \infty$

$$\begin{aligned} \nabla U_0(\mathfrak{x}) &= \frac{\partial}{\partial x'}\left(\nabla_{\mathfrak{x}} \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|}\right) \\ &= (i\varkappa)^2 e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|} \frac{\mathfrak{x}-\mathfrak{x}'}{|\mathfrak{x}-\mathfrak{x}'|^3}(x'-x) + O\left(\frac{1}{r^2}\right). \end{aligned} \tag{3.26}$$

Aus (3.25), (3.26) und (3.19) folgt

$$\nabla U_0(r\mathfrak{x}_0) = i\varkappa\, U(r\mathfrak{x}_0)\,\mathfrak{x}_0 + O\left(\frac{1}{r^2}\right). \tag{3.27}$$

Multipliziert man (3.27) skalar mit $\mathfrak{x}_0$, so erhält man (3.15). Die Gültigkeit von (3.14) folgt unmittelbar aus (3.25). Für $|\mathfrak{x}'| < C$ gelten die Abschätzungen gleichmäßig in $\mathfrak{x}_0$ und $\mathfrak{x}'$. Insgesamt haben wir bewiesen:

Lemma 1. *Für $|\mathfrak{x}'| < C$ erfüllen die Funktionen*

$$U(\mathfrak{x}) = \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|} \quad \textit{und} \quad U_0(\mathfrak{x}) = \frac{\partial}{\partial \mathfrak{n}_{\mathfrak{x}'}} U(\mathfrak{x})$$

die Bedingungen (3.14) *und* (3.15) *gleichmäßig in $\mathfrak{x}_0$ und $\mathfrak{x}'$.*

Mit Hilfe der Lösungen (3.16) und (3.24) können weitere Lösungen der Schwingungsgleichung (3.2) durch Integrationsprozesse gewonnen werden. Ist etwa F eine stetig differenzierbare reguläre Fläche und sind λ und μ auf F definierte stetige Funktionen, so sind

$$V(\mathfrak{x}) = \int_F \lambda(\mathfrak{x}') \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|}\, dF_{\mathfrak{x}'} \tag{3.28}$$

und

$$W(\mathfrak{x}) = \int_F \mu(\mathfrak{x}') \frac{\partial}{\partial n_{\mathfrak{x}'}} \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|}\, dF_{\mathfrak{x}'} \tag{3.29}$$

für $\mathfrak{x} \notin F$ Lösungen von (3.2). $V(\mathfrak{x})$ und $W(\mathfrak{x})$ beschreiben Wellenfelder, die durch eine kontinuierliche Verteilung von einfachen und doppelten Quellen auf F erzeugt werden. Nach Lemma 1 erfüllen $V(\mathfrak{x})$ und $W(\mathfrak{x})$ die Ausstrahlungsbedingung.

Für die physikalische Interpretation der Ausstrahlungsbedingung ist es von Bedeutung, daß in gewissem Sinn auch die Umkehrung der letzten Aussage gilt. Hierzu beweisen wir

Lemma 2. *Es sei F eine stetig differenzierbare, reguläre, geschlossene Fläche mit dem Außenraum G_a und $U(\mathfrak{x})$ eine die Ausstrahlungsbedingung erfüllende, in $G_a + F$ stetig differenzierbare Lösung der Gleichung $\Delta U + \varkappa^2 U = 0$. Dann gilt für $\mathfrak{x} \in G_a$*

$$U(\mathfrak{x}) = \frac{1}{2} \int_F \left[U(\mathfrak{x}') \frac{\partial}{\partial n_{\mathfrak{x}'}} Q(\mathfrak{x}, \mathfrak{x}') - \frac{\partial}{\partial n} U(\mathfrak{x}') \, Q(\mathfrak{x}, \mathfrak{x}') \right] dF_{\mathfrak{x}'}$$

mit

$$Q(\mathfrak{x}, \mathfrak{x}') = \frac{1}{2\pi} \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|}.$$

Zum Beweis betrachten wir einen Punkt $\mathfrak{x}$ aus G_a. Es sei σ_ϱ eine zu F punktfremde Kugel um $\mathfrak{x}$ mit dem Radius ϱ und Σ_r eine σ_ϱ und F umfassende Kugel um den Nullpunkt mit dem Radius r. Ist G das von σ_ϱ, Σ_r und F eingeschlossene Gebiet, so gilt

$$\int_{\Sigma_r - F - \sigma_\varrho} \left(Q \frac{\partial U}{\partial n} - U \frac{\partial Q}{\partial n} \right) dF_{\mathfrak{x}'} = \int_G (Q \, \Delta U - U \, \Delta Q) \, dV_{\mathfrak{x}'} = 0.$$

Aus der Ausstrahlungsbedingung, die von U nach Voraussetzung und von Q nach Lemma 1 erfüllt wird, folgt

$$\lim_{r\to\infty} \int_{\Sigma_r} \left(Q \frac{\partial U}{\partial r} - U \frac{\partial Q}{\partial r} \right) dF_{\mathfrak{x}'} = 0,$$

also

$$\int_F \left(Q \frac{\partial U}{\partial n} - U \frac{\partial Q}{\partial n} \right) dF_{\mathfrak{x}'} = \lim_{\varrho\to 0} \int_{\sigma_\varrho} \left(U \frac{\partial Q}{\partial \varrho} - Q \frac{\partial U}{\partial \varrho} \right) dF_{\mathfrak{x}'} = -2U(\mathfrak{x}).$$

Damit ist Lemma 2 bewiesen.

Lemma 2 besagt, daß sich jede Funktion U, die im Äußeren von F der Schwingungsgleichung (3.2) und der Ausstrahlungsbedingung genügt, als Summe zweier Flächenpotentiale (3.28) und (3.29) darstellen läßt.

Wir wollen noch einige einfache Folgerungen aus Lemma 2 anführen und beweisen hierzu

Lemma 3. *Die Funktion $U(\mathfrak{x})$ genüge außerhalb der zweimal stetig differenzierbaren, regulären, geschlossenen Fläche F der Gleichung $\Delta U + \varkappa^2 U = 0$ und der Ausstrahlungsbedingung. Dann gelten für festes $\mathfrak{a}$ und $r \to \infty$ die asymptotischen Relationen*

$$U(r\mathfrak{x}_0 - \mathfrak{a}) = O\left(\frac{1}{r}\right), \tag{3.30}$$

$$\frac{\partial}{\partial r} U(r\mathfrak{x}_0 - \mathfrak{a}) - i\varkappa \, U(r\mathfrak{x}_0 - \mathfrak{a}) = o\left(\frac{1}{r}\right), \tag{3.31}$$

$$\nabla U(r\mathfrak{x}_0) - i\varkappa \, \mathfrak{x}_0 \, U(r\mathfrak{x}_0) = o\left(\frac{1}{r}\right) \tag{3.32}$$

gleichmäßig in $\mathfrak{x}_0$.

Aus (3.30) und (3.31) folgt, daß die Ausstrahlungsbedingung unabhängig von der Wahl des Nullpunktes ist. (3.32) besagt, daß sich die Radialkomponente von ∇U für $r \to \infty$ wie $O(1/r)$ verhält, während die übrigen Komponenten von ∇U mit der Ordnung $o(1/r)$ verschwinden. Im Sinn der physikalischen Interpretation von ∇U bzw. $\nabla \Phi$ folgt hieraus nach (1.46), daß die Flüssigkeitsteilchen für großes r nahezu in radialer Richtung zum Störzentrum schwingen. Zum Beweis von Lemma 3 beachte man, daß die Flächenpotentiale (3.28) und (3.29) nach Lemma 1 sowie (3.20) und (3.27) die in Lemma 3 formulierten asymptotischen Relationen erfüllen. Aus Lemma 2 folgt daher unmittelbar die Behauptung von Lemma 3.

Als weitere Folgerung aus Lemma 2 sei noch bemerkt, daß für komplexes $\varkappa$ mit $\operatorname{Im}(\varkappa) > 0$ die Ausstrahlungsbedingung mit der Bedingung $U(r\,\mathfrak{x}_0) = o(1/r)$ für $r \to \infty$ äquivalent ist.

Neben den Flächenpotentialen (3.28) und (3.29) können auch räumliche Verteilungen von Punktquellen untersucht werden. Es sei G ein endliches reguläres Gebiet mit der Randfläche F. Dann ist

$$T(\mathfrak{x}) = \int_G \tau(\mathfrak{x}') \frac{e^{i\varkappa|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|} \, dV_{\mathfrak{x}'} \tag{3.33}$$

für $\mathfrak{x} \notin G+F$ eine Lösung von (3.2), die nach Lemma 1 die Ausstrahlungsbedingung erfüllt. Es gilt, wie analog zu dem entsprechenden Satz der Potentialtheorie* gezeigt werden kann,

Lemma 4. *Es sei G ein endliches reguläres Gebiet mit der Randfläche F und $\tau(\mathfrak{x})$ eine in $G+F$ hölderstetige Funktion. Dann ist die durch* (3.3) *definierte Funktion $T(\mathfrak{x})$ überall stetig differenzierbar, und es gilt für $\mathfrak{x} \in G$*

$$\Delta T + \varkappa^2 T = -4\pi\tau.$$

§ 4. Die Eindeutigkeitssätze

Die in den ersten drei Abschnitten dieser Arbeit durchgeführten Untersuchungen ermöglichen es, die Grundaufgaben der Theorie stationärer akustischer Wellenfelder mathematisch zu formulieren. So führt die Berechnung eines durch eine zeitharmonische konservative Kraftdichteverteilung in einem unbegrenzten Medium erzeugten stationären Wellenfelds auf das in der Einleitung formulierte Problem (A). Entsprechend ergeben sich die Probleme (B) und (C), falls in das Medium feste Körper mit der Randfläche F eingebettet sind, die zeitharmonische elastische Schwingungen vollführen, oder falls zwei verschiedene Medien mit der Grenzfläche F' vorliegen.

Wir wollen in diesem Abschnitt zeigen, daß die Probleme (A), (B) und (C) höchstens eine Lösung besitzen. Zur Vereinheitlichung der Beweise gehen wir von folgender Anordnung aus:

Die zu untersuchenden stationären Wellenfelder denken wir uns durch eine räumliche Verteilung zeitharmonischer äußerer Kräfte mit dem Kraftdichtepotential

$$\Psi(\mathfrak{x}, t) = -\frac{c^2}{\omega} \operatorname{Re}\{i f(\mathfrak{x}) e^{-i\omega t}\} \tag{4.1}$$

* Siehe etwa [6], S. 150ff.

sowie durch endlich viele feste Körper $K_1, \ldots, K_n$ mit den stetig differenzierbaren, regulären Randflächen $F_1, \ldots, F_n$ erzeugt, die zeitharmonische elastische Schwingungen mit der Frequenz ω vollführen. Mathematisch drückt sich der Einfluß der Körper K_j durch die Vorgabe einer stetigen Normalgeschwindigkeitsverteilung der Form

$$\frac{1}{\varrho_0(\mathfrak{x})} \operatorname{Re}\{g(\mathfrak{x})\, e^{-i\omega t}\} \tag{4.2}$$

auf den Flächen F_j aus. Zur Abkürzung setzen wir $F_1 + \cdots + F_n = F$. Das Außengebiet G von F werde durch eine zu F punktfremde stetig differenzierbare, reguläre, geschlossene Fläche F' in das Äußere G_a und das Innere G_i zerlegt. Den in G_a liegenden Teil von F bezeichnen wir mit F_a und den in G_i liegenden Teil mit F_i. G_i werde durch ein Medium M_i mit der Dichte $\varrho_{0i}(\mathfrak{x})$ im Ruhezustand, der Schallgeschwindigkeit $c_i(\mathfrak{x})$ und der Dämpfungskonstanten a_i und G_a durch ein Medium M_a mit den entsprechenden Größen $\varrho_{0a}(\mathfrak{x})$, $c_a(\mathfrak{x})$ und a_a ausgefüllt.

Wir setzen voraus, daß die Funktionen ϱ_{0i}, ϱ_{0a} und c_i, c_a in ihren Definitionsbereichen $G_i + F_i + F'$ und $G_a + F_a + F'$ stetig differenzierbar bzw. stetig sind und daß ϱ_{0a} und c_a für $|\mathfrak{x}| > R$ konstant sind. Die Berechnung des durch die Kraftdichteverteilung f und die elastischen Schwingungen der Körper K_j in den Medien M_i und M_a erzeugten stationären Wellenfelds führt zu folgender Aufgabe:

(D) Es ist eine Funktion U mit folgenden Eigenschaften zu bestimmen:

a) U erfüllt in G_i und in G_a die Gleichung

$$\varrho_0 \nabla\left(\frac{1}{\varrho_0} \nabla U\right) + \varkappa^2 U = f$$

mit

$$\varrho_0(\mathfrak{x}) = \begin{cases} \varrho_{0i}(\mathfrak{x}) & \text{für} \quad \mathfrak{x} \in G_i \\ \varrho_{0a}(\mathfrak{x}) & \text{für} \quad \mathfrak{x} \in G_a, \end{cases} \qquad \varkappa(\mathfrak{x}) = \begin{cases} \dfrac{\omega(\omega + i a_i)}{c_i(\mathfrak{x})^2} & \text{für} \quad \mathfrak{x} \in G_i \\ \dfrac{\omega(\omega + i a_a)}{c_a(\mathfrak{x})^2} & \text{für} \quad \mathfrak{x} \in G_a. \end{cases}$$

b) U genügt für $r \to \infty$ der Ausstrahlungsbedingung (3.14), (3.15).

c) U ist in $G_i + F_i + F'$ und in $G_a + F_a + F'$ stetig differenzierbar.

d) Für $\mathfrak{x} \in F'$ gilt

$$(a_a - i\omega)\, U_a + i\,\frac{c_a^2}{\omega} f = (a_i - i\omega)\, U_i + i\,\frac{c_i^2}{\omega} f$$

$$\frac{1}{\varrho_{0a}} \frac{\partial U_a}{\partial n} = \frac{1}{\varrho_{0i}} \frac{\partial U_i}{\partial n}.$$

e) Für $\mathfrak{x} \in F$ gilt

$$\frac{\partial U}{\partial n} = g.$$

Die in der Einleitung formulierten Probleme (A), (B) und (C) gehen aus (D) durch Spezialisierung hervor: (B) erhält man, indem man sich auf Funktionen ϱ_0 und c beschränkt, die im gesamten Äußeren G von F unter Einschluß von F' stetig differenzierbar bzw. stetig sind. (A) und (C) entstehen aus (B) und (D) für $n = 0$ (n = Anzahl der in das Medium eingebetteten festen Körper). Die Eindeutigkeitssätze für die Probleme (A), (B) und (C) folgen somit aus

Satz 1. *Es gibt höchstens eine Funktion U mit den unter* (D) *formulierten Eigenschaften* a) *bis* e).

Zum Beweis nehmen wir an, daß U_1 und U_2 Lösungen des Problems (D) sind. Die Funktion $U = U_1 - U_2$ erfüllt dann die Bedingungen b) und c) sowie

$$\text{a}') \qquad \varrho_0 \nabla\left(\frac{1}{\varrho_0} \nabla U\right) + \varkappa^2 U = 0 \quad \text{für} \quad \mathfrak{x} \in G_i \text{ und } \mathfrak{x} \in G_a,$$

$$\text{d}') \qquad \left.\begin{aligned} (a_a - i\omega)\, U_a &= (a_i - i\omega)\, U_i \\ \frac{1}{\varrho_{0a}} \frac{\partial U_a}{\partial n} &= \frac{1}{\varrho_{0i}} \frac{\partial U_i}{\partial n} \end{aligned}\right\} \quad \text{für} \quad \mathfrak{x} \in F',$$

$$\text{e}') \qquad \frac{\partial U}{\partial n} = 0 \qquad \text{für} \quad \mathfrak{x} \in F.$$

Σ_r sei die Kugel um den Nullpunkt mit dem Radius r. Wir wählen r so groß, daß Σ_r die Flächen F_j und F' umfaßt. Wir betrachten den Ausdruck

$$\operatorname{Im}\left\{(\omega - i a_a) \int\limits_{|\mathfrak{x}|=r} \frac{1}{\varrho_0} \overline{U} \frac{\partial U}{\partial r} dF\right\}, \tag{4.3}$$

der nach (2.20) bis auf den reellen Faktor $\frac{1}{2}$ den mittleren akustischen Energiefluß des Wellenfeldes U durch die Kugel Σ_r darstellt. Beachten wir, daß ϱ_0 und $\varkappa$ für $|\mathfrak{x}| > R$ konstant sind, und setzen wir

$$\varrho_0(\mathfrak{x}) = \varrho_0^*, \qquad \varkappa(\mathfrak{x}) = \varkappa^* \quad \text{für} \quad |\mathfrak{x}| > R, \tag{4.4}$$

so folgt aus der Ausstrahlungsbedingung (3.14), (3.15) für $r \to \infty$

$$\operatorname{Im}\left\{(\omega - i a_a) \int\limits_{|\mathfrak{x}|=r} \frac{1}{\varrho_0} \overline{U} \frac{\partial U}{\partial r} dF\right\} = \frac{1}{\varrho_0^*} \operatorname{Im}\{\varkappa^*(a_a + i\omega)\} \int\limits_{|\mathfrak{x}|=r} |U|^2 dF + o(1). \tag{4.5}$$

Nach (2.6) gilt $0 \leqq \operatorname{arc} \varkappa^* < \pi/4$. Hieraus und aus $\omega > 0$ folgt

$$\operatorname{Im}\{\varkappa^*(a_a + i\omega)\} > 0 \tag{4.6}$$

und somit für $r \to \infty$

$$\operatorname{Im}\left\{(\omega - i a_a) \int\limits_{|\mathfrak{x}|=r} \frac{1}{\varrho_0} \overline{U} \frac{\partial U}{\partial r} dF\right\} \geqq o(1). \tag{4.7}$$

Andererseits gilt, falls wir das von Σ_r, F_a und F' eingeschlossene Teilgebiet von G_a mit G_r bezeichnen,

$$\begin{aligned} &(\omega - i a_a) \int\limits_{|\mathfrak{x}|=r} \frac{1}{\varrho_0} \overline{U} \frac{\partial U}{\partial r} dF \\ &= (\omega - i a_a) \left\{ \int\limits_{G_r} \nabla\left(\frac{1}{\varrho_0} \overline{U} \nabla U\right) dV + \int\limits_{F_a} \frac{1}{\varrho_0} \overline{U} \frac{\partial U}{\partial n} dF + \int\limits_{F'} \frac{1}{\varrho_{0a}} \overline{U}_a \frac{\partial U_a}{\partial n} dF \right\}. \end{aligned} \tag{4.8}$$

Aus a') folgt für $\mathfrak{x} \in G_a$ und $\mathfrak{x} \in G_i$

$$\nabla\left(\frac{1}{\varrho_0} \overline{U} \nabla U\right) = \frac{1}{\varrho_0} |\nabla U|^2 - \frac{\varkappa^2}{\varrho_0} |U|^2. \tag{4.9}$$

Das Integral über F_a verschwindet nach e'). Für $\mathfrak{x}\in F'$ gilt nach d')

$$(4.10)\qquad (\omega - i\,a_a)\,\frac{1}{\varrho_{0a}}\,\overline{U}_a\,\frac{\partial U_a}{\partial n} = (\omega - i\,a_i)\,\frac{1}{\varrho_{0i}}\,\overline{U}_i\,\frac{\partial U_i}{\partial n}\,.$$

Insgesamt erhalten wir damit, falls wir noch (2.5) berücksichtigen,

$$(4.11)\qquad \begin{aligned} (\omega - i\,a_a)\int\limits_{|\mathfrak{x}|=r}\frac{1}{\varrho_0}\,\overline{U}\,\frac{\partial U}{\partial r}\,dF &= (\omega - i\,a_a)\int\limits_{G_r}\frac{1}{\varrho_0}\,|\nabla U|^2\,dV - \\ &- \omega(\omega^2 + a_a^2)\int\limits_{G_r}\frac{1}{\varrho_0\,c_a^2}\,|U|^2\,dV + (\omega - i\,a_i)\int\limits_{F'}\frac{1}{\varrho_{0i}}\,\overline{U}_i\,\frac{\partial U_i}{\partial n}\,dF. \end{aligned}$$

Aus a'), e') und (2.5) folgt

$$\begin{aligned} (\omega - i\,a_i)\int\limits_{F'}\frac{1}{\varrho_{0i}}\,\overline{U}_i\,\frac{\partial U_i}{\partial n}\,dF &= (\omega - i\,a_i)\int\limits_{G_i}\nabla\Big(\frac{1}{\varrho_0}\,\overline{U}\nabla U\Big)\,dV \\ &= (\omega - i\,a_i)\int\limits_{G_i}\Big[\frac{1}{\varrho_0}\,|\nabla U|^2 - \frac{\varkappa^2}{\varrho_0}\,|U|^2\Big]\,dV \\ &= (\omega - i\,a_i)\int\limits_{G_i}\frac{1}{\varrho_0}\,|\nabla U|^2\,dV - \omega(\omega^2 + a_i^2)\int\limits_{G_i}\frac{1}{\varrho_0\,c_i^2}\,|U|^2\,dV \end{aligned}$$

bzw. durch Übergang zum Imaginärteil

$$(4.12)\qquad \operatorname{Im}\Big\{(\omega - i\,a_i)\int\limits_{F'}\frac{1}{\varrho_{0i}}\,\overline{U}_i\,\frac{\partial U_i}{\partial n}\,dF\Big\} = -\,a_i\int\limits_{G_i}\frac{1}{\varrho_0}\,|\nabla U|^2\,dV.$$

Hieraus und aus (4.11) folgt

$$(4.13)\qquad \operatorname{Im}\Big\{(\omega - i\,a_a)\int\limits_{|\mathfrak{x}|=r}\frac{1}{\varrho_0}\,\overline{U}\,\frac{\partial U}{\partial r}\,dF\Big\} = -\,a_a\int\limits_{G_r}\frac{1}{\varrho_0}\,|\nabla U|^2\,dV - a_i\int\limits_{G_i}\frac{1}{\varrho_0}\,|\nabla U|^2\,dV.$$

Da die Dämpfungskonstanten a_a und a_i nicht negativ sind, gilt

$$(4.14)\qquad \operatorname{Im}\Big\{(\omega - i\,a_a)\int\limits_{|\mathfrak{x}|=r}\frac{1}{\varrho_0}\,\overline{U}\,\frac{\partial U}{\partial r}\,dF\Big\} \leqq 0\,,$$

also wegen (4.7) für $r\to\infty$

$$(4.15)\qquad \operatorname{Im}\Big\{(\omega - i\,a_a)\int\limits_{|\mathfrak{x}|=r}\frac{1}{\varrho_0}\,\overline{U}\,\frac{\partial U}{\partial r}\,dF\Big\} = o(1)\,.$$

Hieraus und aus (4.13) folgt

$$(4.16)\qquad a_a\int\limits_{G_a}\frac{1}{\varrho_0}\,|\nabla U|^2\,dV + a_i\int\limits_{G_i}\frac{1}{\varrho_0}\,|\nabla U|^2\,dV = 0\,.$$

Im Fall, daß a_a und a_i von 0 verschieden sind, gilt nach (4.16) für alle $\mathfrak{x}$ aus G_a und G_i $\nabla U = 0$, also auch $\varrho_0\,\nabla\Big(\frac{1}{\varrho_0}\,\nabla U\Big) = 0$ und somit nach a') $U = 0$. Damit ist Satz 1 im Fall $a_a \neq 0$ und $a_i \neq 0$ bewiesen.

Zum Beweis von Satz 1 für $a_a = 0$ benutzen wir zwei Ergebnisse von RELLICH [*13*] und HEINZ [*4*]:

Lemma 5. *Für* $|\mathfrak{x}| > R$ *gelte* $\Delta U + \varkappa^2 U = 0$ *mit konstantem, positivem* $\varkappa$. *Gilt dann für* $r \to \infty$

$$\int_{|\mathfrak{x}|=r} |U|^2 dF = o(1),$$

so verschwindet U *für* $|\mathfrak{x}| > R$.

Lemma 6. *Es sei* G *ein Gebiet mit der stetig differenzierbaren, regulären, geschlossenen Randfläche* F. *Ferner sei* $\mathfrak{x}_0$ *ein Punkt auf* F *und* $F(\mathfrak{x}_0, \delta)$ *der innerhalb der Kugel* $|\mathfrak{x} - \mathfrak{x}_0| = \delta$ *liegende Teil von* F. *Die Funktion* U *sei in* $G + F(\mathfrak{x}_0, \delta)$ *stetig differenzierbar und genüge in* G *der Gleichung* $\varrho_0 \nabla\left(\frac{1}{\varrho_0} \nabla U\right) + \varkappa^2 U = 0$. *Gilt dann für alle Punkte auf* $F(\mathfrak{x}_0, \delta)$ $U = \frac{\partial U}{\partial n} = 0$, *so verschwindet* U *identisch in* G.

Einen elementaren Beweis des zunächst von Rellich [*13*] mit Hilfe der Theorie der Kugelfunktionen bewiesenen Lemmas 5 gibt Miranker [*9*]. Lemma 6 ist in dem Eindeutigkeitssatz für das Cauchysche Anfangswertproblem enthalten, den Heinz [*4*] für Gleichungen der Form

$$\Delta U = F\left(\mathfrak{x}, U, \frac{\partial U}{\partial x_j}\right)$$

bewies und der später von Cordes [*2*] und Aronszajn [*1*] auf den Fall übertragen wurde, daß an die Stelle des Δ-Operators ein beliebiger elliptischer Differentialoperator zweiter Ordnung tritt.

Im Fall $a_a = 0$ folgt für $r \to \infty$ aus (4.15) und der Ausstrahlungsbedingung

$$\int_{|\mathfrak{x}|=r} |U|^2 dF = o(1). \tag{4.17}$$

Nach Lemma 5 verschwindet daher U für $|\mathfrak{x}| > R$ und somit nach Lemma 6 auch für $\mathfrak{x} \in G_a$.

Insbesondere gilt auf F'

$$U_a = \frac{\partial U_a}{\partial n} = 0, \tag{4.18}$$

also nach d′) auch

$$U_i = \frac{\partial U_i}{\partial n} = 0. \tag{4.19}$$

Nach Lemma 6 verschwindet daher U auch in G_i. Damit ist Satz 1 auch im Fall $a_a = 0$ bewiesen.

Es sei noch bemerkt, daß Lemma 5 nach Kato [*5*] auf Gleichungen der Gestalt

$$\Delta U + \varkappa(\mathfrak{x})^2 U = 0 \tag{4.20}$$

ausgedehnt werden kann, wobei $\varkappa(\mathfrak{x})$ für $|\mathfrak{x}| \to \infty$ gegen eine positive Zahl $\varkappa^*$ konvergiert und für beliebiges $\varepsilon > 0$ die Abschätzung

$$\varkappa(\mathfrak{x}) - \varkappa^* = O\left(\frac{1}{|\mathfrak{x}|^{1+\varepsilon}}\right) \tag{4.21}$$

gleichmäßig für alle Richtungen erfüllt ist. Es ist zu vermuten, daß sich die Methoden von Kato unter geeigneten, zu (4.21) analogen Voraussetzungen über ϱ_0 auf die Gleichung

$$\varrho_0 \nabla\left(\frac{1}{\varrho_0} \nabla U\right) + \varkappa(\mathfrak{x})^2 U = 0 \tag{4.22}$$

übertragen lassen. Es scheint daher, daß in den Eindeutigkeitsuntersuchungen dieses Abschnitts und auch in den Existenzuntersuchungen des nächsten Abschnitts die Voraussetzung, daß $\varkappa(\mathfrak{x})$ und $\varrho_0(\mathfrak{x})$ für $|\mathfrak{x}|>R$ konstant sind, abgeschwächt werden kann. Wir wollen jedoch diesen Fragenkreis in der vorliegenden Arbeit nicht weiter verfolgen.

§ 5. Akustische Wellenfelder in unbegrenzten Medien mit stetig veränderlichen Stoffeigenschaften

In diesem Abschnitt wollen wir das erste der in der Einleitung formulierten Probleme behandeln. Wir betrachten ein unbegrenztes Medium mit der Dichte $\varrho_0(\mathfrak{x})$ im Ruhezustand und der Schallgeschwindigkeit $c(\mathfrak{x})$ und setzen voraus, daß ϱ_0 im ganzen Raum hölderstetig differenzierbar und c für alle $\mathfrak{x}$ hölderstetig ist. Für $|\mathfrak{x}|>R$ seien ϱ_0 und c konstant. Wir setzen

$$\varrho_0(\mathfrak{x})=\varrho_0^*, \qquad c(\mathfrak{x})=c^* \quad \text{für} \quad |\mathfrak{x}|>R, \tag{5.1}$$

$$\varkappa(\mathfrak{x})^2=\frac{\omega(\omega+i\,a)}{c(\mathfrak{x})^2}, \qquad \varkappa^{*2}=\frac{\omega(\omega+i\,a)}{c^{*2}}. \tag{5.2}$$

Wir wählen $\varkappa(\mathfrak{x})$ so, daß $0\leqq \operatorname{arc}\varkappa(\mathfrak{x})<\frac{1}{2}\pi$ gilt.

Die Aufgabe, das durch eine zeitharmonische Kraftdichteverteilung mit dem Potential (4.1) erzeugte stationäre akustische Wellenfeld zu berechnen, verlangt die Bestimmung einer Funktion U, die für alle $\mathfrak{x}$ der Gleichung

$$\varrho_0 \nabla\left(\frac{1}{\varrho_0}\nabla U\right)+\varkappa^2 U=f \tag{5.3}$$

sowie der Ausstrahlungsbedingung (3.14), (3.15) mit $\varkappa=\varkappa^*$ genügt (Problem (A)). Wir setzen voraus, daß f hölderstetig ist und für $|\mathfrak{x}|>R$ verschwindet. Nach Satz 1 besitzt Problem (A) höchstens eine stetig differenzierbare Lösung U.

Es sei U eine hölderstetig differenzierbare Funktion mit den verlangten Eigenschaften. Nach (5.3) gilt

$$\Delta U+\varkappa^{*2}U=\begin{cases}(\varkappa^{*2}-\varkappa^2)\,U-\varrho_0\nabla\dfrac{1}{\varrho_0}\nabla U+f & \text{für} \quad |\mathfrak{x}|<R\\[2mm] 0 & \text{für} \quad |\mathfrak{x}|\geqq R.\end{cases} \tag{5.4}$$

Da nach Voraussetzung $\varkappa$ und f hölderstetig sowie ϱ_0 und U hölderstetig differenzierbar sind, ist

$$(\varkappa^{*2}-\varkappa^2)\,U-\varrho_0\nabla\frac{1}{\varrho_0}\nabla U+f$$

hölderstetig. Hieraus sowie aus Lemma 4 und (5.4) folgt, daß die Funktion

$$V=U+\frac{1}{4\pi}\int\limits_{|\mathfrak{x}'|<R}\left[(\varkappa^{*2}-\varkappa^2)\,U-\varrho_0\nabla\frac{1}{\varrho_0}\nabla U+f\right]\frac{e^{i\varkappa^*|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|}\,dV_{\mathfrak{x}'} \tag{5.5}$$

im gesamten Raum der Gleichung $\Delta V+\varkappa^{*2}V=0$ und der Ausstrahlungsbedingung genügt. Aus Satz 1 folgt daher $V\equiv 0$ bzw.

$$U(\mathfrak{x})=-\frac{1}{4\pi}\int\limits_{|\mathfrak{x}'|<R}\tau(\mathfrak{x}')\,\frac{e^{i\varkappa^*|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|}\,dV_{\mathfrak{x}'} \tag{5.6}$$

mit

$$\tau = (\varkappa^{*2} - \varkappa^2)\, U - \varrho_0 \nabla \frac{1}{\varrho_0} \nabla U + f. \tag{5.7}$$

Da τ hölderstetig ist, gilt nach Lemma 4 und (5.3) für $|\mathfrak{x}| < R$

$$\begin{aligned}\varrho_0 \nabla\left(\frac{1}{\varrho_0} \nabla U\right) + \varkappa^2 U &= \Delta U + \varkappa^{*2} U + (\varkappa^2 - \varkappa^{*2})\, U + \varrho_0 \nabla \frac{1}{\varrho_0} \nabla U \\ &= \tau(\mathfrak{x}) + \frac{1}{4\pi} \int\limits_{|\mathfrak{x}'|<R} \left\{ [\varkappa^{*2} - \varkappa(\mathfrak{x})^2]\, \frac{e^{i\varkappa^*|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|} \right. \\ &\qquad \left. - \varrho_0(\mathfrak{x})\, \nabla \frac{1}{\varrho_0(\mathfrak{x})} \nabla_{\mathfrak{x}} \frac{e^{i\varkappa^*|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|} \right\} \tau(\mathfrak{x}')\, dV_{\mathfrak{x}'} = f(\mathfrak{x}).\end{aligned}$$

Damit ist gezeigt: Ist U eine hölderstetig differenzierbare Funktion, die der Gleichung (5.3) und der Ausstrahlungsbedingung genügt, so gibt es eine hölderstetige Funktion τ, mit deren Hilfe sich U in der Form (5.6) darstellen läßt; τ genügt für $|\mathfrak{x}| < R$ der Integralgleichung

$$\tau(\mathfrak{x}) - \int\limits_{|\mathfrak{x}'|<R} \tau(\mathfrak{x}')\, K(\mathfrak{x}, \mathfrak{x}')\, dV_{\mathfrak{x}'} = f(\mathfrak{x}) \tag{5.8}$$

mit

$$K(\mathfrak{x}, \mathfrak{x}') = \frac{1}{4\pi} \left\{ [\varkappa(\mathfrak{x})^2 - \varkappa^{*2}]\, \frac{e^{i\varkappa^*|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|} + \varrho_0(\mathfrak{x})\, \nabla \frac{1}{\varrho_0(\mathfrak{x})} \nabla_{\mathfrak{x}} \frac{e^{i\varkappa^*|\mathfrak{x}-\mathfrak{x}'|}}{|\mathfrak{x}-\mathfrak{x}'|} \right\}. \tag{5.9}$$

Zur Konstruktion einer Lösung U von Gleichung (5.3) bei gegebenem ϱ_0, $\varkappa$ und f diskutieren wir die Integralgleichung (5.8). Die Zuordnung

$$\tau(\mathfrak{x}) \to K\tau(\mathfrak{x}) = \int\limits_{|\mathfrak{x}'|<R} \tau(\mathfrak{x}')\, K(\mathfrak{x}, \mathfrak{x}')\, dV_{\mathfrak{x}'} \tag{5.10}$$

kann als eine lineare Transformation im Banach-Raum C_R der für $|\mathfrak{x}| \leqq R$ stetigen Funktionen τ aufgefaßt werden*. Die Norm in C_R wird hierbei durch

$$\|\tau\| = \max_{|\mathfrak{x}| \leqq R} |\tau(\mathfrak{x})| \tag{5.11}$$

erklärt. Wir zeigen, daß die Transformation K beschränkt und vollstetig ist. Hierzu beweisen wir

Lemma 7. *Die Funktion $\tau(\mathfrak{x})$ sei für $|\mathfrak{x}| \leqq R$ stetig. Ist $\|\tau\|$ das Maximum von $|\tau(\mathfrak{x})|$ in $|\mathfrak{x}| \leqq R$, so gibt es eine von $\mathfrak{x}$ und τ unabhängige positive Zahl C, so daß für alle $\mathfrak{x}$ mit $|\mathfrak{x}| \leqq R$*

$$|K\tau(\mathfrak{x})| \leqq C\, \|\tau\| \tag{5.12}$$

sowie für alle $\mathfrak{x}_1$, $\mathfrak{x}_2$ mit $|\mathfrak{x}_1| \leqq R$, $|\mathfrak{x}_2| \leqq R$ und $|\mathfrak{x}_1 - \mathfrak{x}_2| \leqq 1$

$$|K\tau(\mathfrak{x}_1) - K\tau(\mathfrak{x}_2)| \leqq C\, \|\tau\| \cdot |\mathfrak{x}_1 - \mathfrak{x}_2|^{\frac{1}{2}} \tag{5.13}$$

gilt.

Beweis. Nach (5.9) gibt es eine Konstante A, so daß für alle $\mathfrak{x}, \mathfrak{x}'$ mit $|\mathfrak{x}| \leqq R+2$ und $|\mathfrak{x}'| \leqq R+2$

$$|K(\mathfrak{x}, \mathfrak{x}')| \leqq \frac{A}{|\mathfrak{x}-\mathfrak{x}'|^2} \tag{5.14}$$

und

$$|\nabla_{\mathfrak{x}} K(\mathfrak{x}, \mathfrak{x}')| \leqq \frac{A}{|\mathfrak{x}-\mathfrak{x}'|^3} \tag{5.15}$$

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* Zur Theorie der Banach-Räume siehe etwa [*14*].

Satz 2. *Das in der Einleitung formulierte Problem* (A) *besitzt unter den zu Beginn dieses Abschnitts gemachten Voraussetzungen über* ϱ_0, c *und* f *genau eine stetig differenzierbare Lösung* U. *Man erhält* U *durch* (5.26), *wobei* τ *die eindeutig bestimmte Lösung der Integralgleichung* (5.8), (5.9) *ist.*

Zur praktischen Berechnung von U können nach Satz 2 die numerischen Methoden zur Auflösung Fredholmscher Integralgleichungen, etwa das Verfahren von CL. MÜLLER [*10*], verwendet werden.

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